Which Logic for the Radical antirealist?

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Abstract

Since the ground-breaking contributions of M. Dummett (Dummett 1978), it is widely recognized that antirealist principles have a critical impact on the choice of logic. According to Dummett, classical logic does not satisfy the requirements of antirealism, but *intuitionistic logic* does. Some philosophers have adopted a more radical stance and argued for a more important departure from classical logic on the basis of similar intuitions. In particular, J. Dubucs and M. Marion (see (Dubucs & Marion 2003) and (Dubucs 2002)) have recently argued that a proper understanding of antirealism should lead us to the so-called *substructural logics* (see (Restall 2000)) and especially linear logic. The aim of this paper is to scrutinize this proposal. We will raise two kinds of issues for the radical antirealist. First, we will stress the fact that it is hard to live without structural rules. Second, we will argue that, from an antirealist perspective, there is currently no satisfactory justification to the shift to substructural logics.

Introduction

One of the most striking outcome of the controversy between semantic realism and semantic antirealism concerns logic: it is widely held that an antirealist position should result in a *revisionist* attitude towards classical logic. More precisely, according to the seminal contributions of Michael Dummett, a coherent antirealist should prefer intuitionistic logic to classical logic. Therefore, one would expect that different forms of antirealism would result in different forms of revisionism. Recently, it has been argued by J. Dubucs and M. Marion (Dubucs & Marion 2003) that an antirealism more radical

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than the usual one justifies another logic than the intuitionistic, namely *linear logic* (Girard 1987). If one calls 'moderate" the version of antirealism advocated by Dummett, the landscape is going to be the following:

realism	$\Rightarrow$	classical logic
moderate antirealism	$\Rightarrow$	intuitionistic logic
radical antirealism	$\Rightarrow$	linear logic

The aim of this paper is not to take a position on the left-hand side of the tabular: we take position neither in the realism/antirealism debate nor in the moderate/radical antirealism debate. The precise target on which we focus is the idea that a strengthening of the moderate antirealist's basic insights leads to linear logic rather than to intuitionistic logic. In other words, we wonder whether there is a path from the bottom-left cell to the bottom-right cell that is parallel to the usual path that goes from the middle-left cell to the middle-right.

We will proceed as follows. In section 1, we give a rough reconstruction of antirealism's basic tenets and of the substructural revisionism of the so-called radical antirealists. We then scrutinize both the consequences of committing to substructural revisionism and the principles that could this commitment. In section 2, we are argue that, because of the splitting of connectives, it is not easy to live without structural rules. Therefore the justification for such a shift has to be pretty firm. But in section 3, we show that there is currently no satisfactory foundation for substructural revisionism. In section 4, nonetheless, we sketch of a possible, game-theoretic, way to achieve such a foundation.

1 From antirealism to substructural logic

1.1 Moderate antirealism

We shall first reconstruct briefly the position of *moderate* antirealism. Though we do not want to enter into an exegetical discussion, the view we present here could be called Dummettian antirealism as well. We take moderate antirealism to consist in two basic components: the antirealist component *per se* and the revisionist component.

Moderate antirealism starts with a rejection of truth-conditional semantics. According to truth-conditional semantics, the meaning of a declarative sentence S is the condition under which it is true – and to grasp the meaning of a sentence S is to grasp its truth-conditions. Furthermore, the truth or falsity of S is independent of our means of knowing it[1]: nothing precludes that the conditions under which S is true cannot be recognized as such when they obtain. To put it another way, S could be true though it is not possible to know that it is true. According to the realist, truth in not epistemically constrained.

The antirealist rejects precisely this lack of epistemic constraint: if a sentence S is true, then it should be possible to recognize that it is true. (this is the so-called "Knowability Principle".) There are two main arguments in favor of the knowability principle: if knowledge of meaning is to be analyzed as knowledge of truth-conditions, one has to be able to gain such knowledge (this is the *learnability argument*) and to manifest that one possesses it (this is the *manifestability argument*). As long as truth-conditions are recognition-transcendent, knowledge of truth-conditions does not satisfy these requirements; and the realist thus fails to account for our mastery of language.[2]

As a consequence, the antirealist rejects the notion of truth-conditions as an adequate basis for a theory of meaning, and puts forward as an alternative the "conditions under which we acknowledge the statement as conclusively established"[3] or, as it is sometimes put, assertibility-conditions, *i.e.* the conditions under which one is justified to assert the sentence.[4] In the case of mathematical discourse, one is justified to assert a sentence just in case one has a proof of that sentence. Therefore, the meaning of a mathematical sentence consists in its provability conditions (as opposed to its mysterious recognition transcendent truth-conditions).[5]

[1]See (Dummett 1978), "Realism", p. 146: "Realism I characterize as the belief that statements of the disputed class possess an objective truth-value, independently of our means of knowing it..."

[2]In the case of mathematical discourse, see (Dummett 1978), "The Philosophical Basis of Intuitionistic Logic", p.225: "If to know the meaning of a mathematical statement is to grasp its use; if we learn the meaning by learning its use, and our knowledge of its meaning is a knowledge which we must be capable of manifesting by the use we make of it: then the notion of truth, considered as a feature which each mathematical determinately possesses or determinately lacks, independently of ours means of recognizing its truth-value, cannot be the central notion for a theory of the meanings of mathematical statements..."

[3](Dummett 1978), "The Philosophical Basis of Intuitionistic Logic", p.226.

[4]See (Dummett 1978), "The Philosophical Basis of Intuitionistic Logic", p.225: "We must, therefore, replace the notion of truth, as the central notion of the theory of meaning for mathematical statements, by the notion of *proof*: a grasp of the meaning of a statement consists in a capacity to recognize a proof when one is presented to us..."

[5]See (Dummett 1978), "The Philosophical Basis of Intuitionistic Logic", pp. 226-7 and (Prawitz 1977), p. 20.

This is it for the basic tenets of moderate antirealisms. What we want to stress now is that (and how) *these tenets lead to logical revisionism*: they give us strong reasons to reject classical logic and, at least as far as mathematical discourse is concerned, to endorse intuitionistic logic. As a matter of fact, the path from antirealism to logical revisionism is not as clear as one might wish. Actually, we think that there are *two ways* from antirealism to logical revisionism: one that goes directly from the rejection of realism to the rejection of the law of excluded middle; and one that goes through proof-theoretic arguments from the endorsement of antirealism to a thorough justification of intuitionistic logic[6] We shall call the first way *high-level revisionism* and the second *low-level revisionism.*

Let us elaborate on this distinction, which will play an important role in our discussion of radical antirealism. *High-level revisionism* consists in the rejection of the excluded middle which ensues directly from the acceptance of the knowability principle. More precisely, for Dummett, the rejection of the law of excluded middle (LEM for short) rests upon the rejection of the principle of bivalence according to which every (meaningful, non-vague and non-ambiguous) declarative sentence is determinately true or false. Although the principle of bivalence is not equivalent to LEM, but, as Dummett puts it, "once we have lost any reason to assume every statement to be either true or false, we have no reason, either, to maintain the law of excluded middle" ((Dummett 1991), p.9)

The exact argument for the rejection of bivalence is a matter of controversy[7]. J. Salerno has convincingly argued that Dummett's and Wright's (see Wright 1992, p. 43) arguments are unsound, but he has also proposed an amended version. His point is that the following three are incompatible:

(i) It is known that LEM holds.

[6]See (Read 1995) for a closely related presentation of the antirealist case for revisionism: what we call "high-level revisionism" corresponds roughly to what Read calls the "Linguistic Argument" and what we call low-level revisionism corresponds to what he calls the "Logical Argument". The distinction is implicit in other places: e.g. in Tennant's (Tennant 1997), chapters 6-7 focus on high-level revisionism, whereas chapter 10 focuses on low-level revisionism.

[7]The question whether dummettian antirealism justifies rightly revisionism has been (and is) much disputed. See C. Wright, "antirealism and revisionism" in (Wright 1993), (Tennant 1997), (Salerno 2000), (Cogburn 2002), (Cogburn 2003). In particular, (Tennant 1997) argues that Dummett's manifestation argument, even if it is an "attempted reductio of the principle of bivalence", *in so far as it is directed against bivalence*, is, when properly regimented, revealed as embodying "a non-sequitur of numbing grossness".

(ii) The Knowability Principle is known: We know that for all A, if A is true, then it is possible to prove that A.

(iii) We do not know that for all A, either it is possible to prove A or it is possible to prove the negation of A. This is a principle of epistemic modesty.

We have just briefly recalled the arguments in favor of the Knowability Principle. The principle of epistemic modesty is reasonable as well: there are large classes of sentences for which we do not possess any decision procedure, where by a decision procedure, we mean an effective method yielding a proof of A if A holds and a proof of the negation of A, if the negation of A holds. It follows then that we should reject the first claim, namely that we are entitled to assert LEM in full generality.

On the contrary, it is clear that the LEM does hold for those classes of sentences for which we do possess a method for deciding them. But the argument above shows that it is not sound in general to assume LEM, and that we should hold on to it *only when we are concerned with decidable classes of sentences*. The antirealist is therefore a logical revisionist in so far as she draws a line between those statements for which LEM can be asserted and those for which it cannot. And decidability is the criterion used to draw this line, because decidability is both necessary and sufficient for us to be entitled to assert LEM.

Without entering into the details of Salerno's argument, we shall be content with this presentation of high-level revisionism. Let us consider now what we have called *low-level revisionism*. There is a normative component that any theory of meaning based on assertibility conditions should abide by: What can be inferred from a given sentence should not go beyond what is required in order to be entitled to assert it. This principle of *harmony* takes a precise form in the setting of natural deduction which is used to provide the meaning of logical and mathematical expressions.[8] In natural deduction, the assertability conditions – the conditions for being in position of asserting

[8]Natural deduction can come either as a mono-conclusion system or as a multiple conclusion system. Harmony rules out classical logic only if the system admits only of a single conclusions at a time. Dummett has argued that using multiple conclusions is not ok because this presupposes an (unsound) classical understanding of disjunction. This point has been recently challenged by Restall (see (Restall 2005)). Restall proposes a conceptual foundation of a system with multiple conclusions based on two primitives, assertion *and* denial. This seems to us to be a very promising response to the antirealist challenge againt classical logic, though a discussion of Restall's arguments would lead us beyond the scope of this paper.

a statement – are given by the introduction rules, and the corresponding "exploitability conditions" – what can be inferred from a statement – are given by the elimination rules. The principle of harmony has it that every detour consisting in an introduction rule followed by an elimination rule for the same expression should be eliminable.

It turns out that the rules of intuitionistic logic satisfy harmony. But, under the assumption that the calculus should not allow for multiple conclusions, LEM or other principles yielding classical logic like double negation elimination cannot be added in such a way that harmony obtains. We should therefore reject classical logic in favor intuitionistic logic, because the latter, but not the former, is satisfactory from a normative perspective. Low-level revisionism is thus based on a proof-theoretic semantics. It is important to note that this is a two-stage path: first, one endorses an assertibilty-conditions theory of meaning; then, as a by-product, classical logic is disqualified and intuitionistic logic is justified.

A striking feature of logical revisionism along the lines of moderate antirealism emerges when one compares high-level revisionism with low-level revisionism: both lead to the very same conclusion, namely that LEM should be rejected. On the one hand, high-level revisionism discards the principle of bivalence, leaving with no reason to accept LEM. On the other hand, low-level revisionism justifies intuitionistic logic, which may be construed as classical logic minus LEM. Not only low-level revisionism is consistent with high-level revisionism, but it does not advocate any further departure from classical logic than the one which is required by high-level revisionism. There is thus some kind of "meta-harmony" betweeen the two levels of revisionism. As Dummett puts it ((Dummett 1993), p.75), "A theory of meaning in terms of verification is bound to yield a notion of truth for which bivalence fails to hold for many sentences which we are unreflectively disposed to interpret in a realistic manner". Low-level revisionism shows that a theory of meaning in terms of verification does yield the logic it is bound to yield on account of high-level revisionism.

1.2 Radical antirealism

The radical antirealist shares the basic tenets of the antirealist, but she thinks her colleague is too shy when it comes to putting epistemic constraints on truth. As a consequence, the radical antirealist will be a revisionist too, but she will be an even more radical one. We will start by explaining how and why the basic principles of antirealism are radicalized, and then we will examine the consequence of this move for logic.

1.2.1 Decidability in principle and decidability in practice

According to the antirealist, if a statement is true, one has to be able to recognize that it is true. And for LEM to hold, statements have to be decidable. But what does it mean to say that one has to be able to recognize that something is true, or to decide if a statement is provable or disprovable ? On the one hand, the cognitive abilities of a not that gifted sophomore are certainly not the absolute norm by which truth should be constrained. On the other hand, the limitless powers of the divine intellect are not a reasonable candidate either: if truth is only constrained by what God can do, and if God can do anything, this is a cheap constraint indeed.

What are the norms by which recognizability of truth are to measured? The moderate antirealism does not choose God's point of view; indeed Dummett acknowledges that if these norms were taken to be those of God, realism and antirealism would conflate into one and the same position. However, moderate antirealism is still quite *liberal* with respect to these epistemic constraints: for a set of sentences to be a decidable class, it is only required that such sentences might be decidable *in principle* by a creature with a finite mind, that is by finitary procedures.

Now, the problem is that moderate antirealism has to face some kind of revenge. If truth has to be epistemically constraint in order to satisfy manifestability requirements, these contrainsts has to be strong enough to guarantee that knowledge of truth is manifestable. But think of a set of sentences which is decidable in principle, but such that the truth or falsity of each sentence can only be established by methods which are *practically* out of reach. In that case, what is there to be exhibited? If the decision procedure cannot actually be used and applied, in which sense would knowledge of these methods be any more human than God's knowledge? What would it mean to manifest such a knowledge, or to be able to acquire it? Thus it seems that for such a set of sentence the moderate antirealist fails to satisfy the requirements that she has herself advertised against realism. Granting this point, moderate antirealism appears as an *unstable* position: epistemic constraints on truth might be discarded right at the beginning, but if there are such constraints, they should be taken seriously and they should be measured by decidability *in practice* instead of decidability *in principle*.

1.2.2 The radical antirealist crush on substructural logic

There have been various attempts to implement this radicalization, among which strict finitism is one of the most famous (as elaborated for example in (Wright 1993)). In this paper however, we shall focus on another version of radical antirealism recently advocated in (Dubucs & Marion 2003).[9] According to Dubucs and Marion, the outcome of the radical antirealist revision procedure should no longer be intuitionistic logic. They claim that substructural logics, and more precisely linear logic, are more faithful to the basic insights of antirealism than intuitionistic logic.

Let us see why. In standard presentations of sequent calculus, different types of rules are distinguished: there are on the one hand the *logical rule*, which makes for the introduction of logical connectives, and there are on the other hand the *structural rules*, like the rules of Weakening and Contraction, which correspond to properties of the consequence relation itself.[10] Here are Weakening and Contraction:

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\text{ Weakening} \qquad \frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\text{ Contraction}$$

The radical antirealist's idea is that some substructural rules contain crucial elements of epistemic idealization. Hence, in order to "unidealize" logic from an epistemic point of view, one should control these structural rules. A new logical revisionism follows: the claim is now that a substructural logic like linear logic is justified from an antirealistic point of view.

This is the radical antirealist crush on substructural logic, and the aim of this paper is to evaluate it. One may basically evaluate such a proposal from two points of view: from the point of view of the *principles* that could lead to it and from the point of the *consequences* that would result from its endorsement. We will proceed to the evaluation from both points of view and deal with the following two questions:

- How can one live without them?
- Why should one divorce from the structural rules?

[9] Explaining in details why we favor this approach over strict finitism would lead us beyond the scope of this paper. Basically, we agree with the arguments by Dubucs and Marion against strict finitism: specifying by brute force what it means to be feasible – say it means "being doable in less than n steps of computation", or "doable in a reasonably small number of steps" – is bound to lead to soritic paradoxes. Despite the criticisms that we develop on here, we take the proposal by Dubucs and Marion to be the most attractive one among various versions of radical antirealism, precisely because it aims at getting a non stipulatory grip on feasibility.

[10] Note that this distinction is not tied to the adoption of sequent calculus as a proof system. A similar point could be made using say natural deduction, tableaux methods or a dialogical setting. Arguably, any good framework for proofs should be able to distinguish between abstract properties of the consequence relation, that may or may not be used in proofs, and the mere characterization of logical connectives by logical rules.

2 Life without the structural rules

The opponents to semantic antirealism have always been prompt to notice that there is something paradoxical in the antirealist's position: the antirealist bases his rejection of realism on slogans such as "meaning is use" and she ends up with a proposal to revise usage. Stated in polemical terms, this amounts to saying that the antirealist's attitude towards use is opportunist: she invokes use when it is useful to do so and repudiates it when needed. antirealists grant the existence of such a tension, but claim that there is nothing preposterous in it. However, M. Dummett admits that the greater the revisions, the less plausible the theory, because "the principal purpose of a theory of meaning is to explain existing practice rather than to criticize it."[11]

Obviously, this tension will be all the more vivid in the context of radical antirealism. If its advocate grants with Dummett that an increase in departure from "existing linguistic practice" yields a decrease in the theory's plausibility, then she cannot but hope that the shift to substructural logic is not a too dramatic revision.

How should we assess the acceptability of a revisionist proposal with respect to standard use? In the case of logical connectives, we take it that the most elementary inferences that speakers accept as part of a characterization of what these connectives mean should be recognized as valid. Of course what these inferences are is a matter of debate, but we shall argue that, on any account of what are the basic meaning-constitutive inferences for the logical connectives, the revision in point is quite severe. Our concern is related to a one well-known feature of substructural logics, namely the so-called phenomenon of splitting of logical connectives. Let us consider these two pairs of rules for conjunction in (intuitionistic) sequent calculus:

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C} \qquad\qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C} \qquad\qquad \frac{\Gamma_1 \vdash A \qquad \Gamma_2 \vdash B}{\Gamma_1, \Gamma_2 \vdash A \wedge B}$$

One can check that the two pairs of rules are equivalent, in the sense that each one can be derived from the other. But this derivation resorts crucially to the structural rules of Weakening and Contraction. Without such rules, the equivalence does not hold. Therefore, in a context in which the structural rules are not valid, one gets two different conjunctive connectives: one that

[11] (Dummett 1993), "What is a Theory of Meaning? (II)", p.75.

corresponds to the first pair of rules, and the other that corresponds to the second pair of rules. The latter is called *fusion* in the relevant logic literature, *multiplicative conjunction* or *times* (notation $\otimes$) in the linear logic literature. The former is called *additive conjunction* or *with* (notation &) in the linear logic literature.

The two right-introduction rules make the difference between the two conjunctions salient: in the additive case, there is one antecedent Γ which is common to $\Gamma \vdash A$ and $\Gamma \vdash B$, whereas in the multiplicative case, the antecedents may be different. For our discussion, the main question is to know what are the connections between these two connectives and our pre-theoretical notion of conjunction. Let us consider the two connectives in turn:

(i) $\otimes$: one can easily show that the following sequent is derivable:

$$A, B \vdash A \otimes B$$

which seems to be a highly desirable feature for a conjunction: to get A and B, I just need both A and B. Furthermore, the interaction between $\otimes$ and $\rightarrow$ satisfies the so-called residuation property:

$$A \rightarrow (B \rightarrow C) \equiv (A \otimes B) \rightarrow C$$

If A implies that B implies C, then I can get C from A and B, and *vice versa*. That's pretty much reasonable. But note that on the contrary sequents of the form

$$A \otimes B \vdash A$$

are *not* derivable. From a pre-theoretical point of view, this behavior of $\otimes$ is weird. If I can show that A and B, why should not be able to assert A?

(ii) &: the additive conjunction has welcome features as well, since, for instance, the following sequent is derivable:

$$A \& B \vdash A$$

But this time, it is no longer possible to derive the following one:

$$A, B \vdash A \& B$$

To put it bluntly, multiplicative conjunction seems to describe nicely the conditions under which a conjunction can be asserted but not what can be inferred from a conjunction. On the contrary, additive conjunction seems to describe nicely what can be inferred from a conjunction but not the conditions under which a conjunction can be asserted.

Several replies are available to the radical antirealist. She can argue that one of two connectives is the true one, but given the properties described above, this does not seem very plausible. Another reply would consist to bite the bullet and consider that linear logic refines on our pre-theoretic use of conjunction which is ambiguous. From this point of view, contrary to what the layman thinks, there is no single well-justified conjunction. The layman might be wrong, in the sense that our best theory of meaning might have among its consequences that "and" is indeed ambiguous. However, note that "and" fails the standard linguistic test for ambiguity, namely cross-linguistic disambiguation. "Bank" in English is ambiguous between some place where I can get money and some slop beside a river where I can sit and wait for the fish to bite the hook. One good reason to think that there are two different lexical entries for "bank" is that in other languages, like French, there are two different words for that, namely "banque" and "rive". We do not know of any spoken language in which there would be two different words for "and", one corresponding to the additive conjunction, the other one to the multiplicative conjunction.

Whatever the reply the radical antirealist chooses, we take this to show that life without structural rules is not easy. Arguably, this does not constitute a knockdown argument to reject the radical antirealist's proposal. But given these difficulties, there has at least to be very good reasons to divorce from structural rules. In other words, the reasons for rejecting structural rules have to be pretty strong in order to balance the cost of living without them. Hence, we now turn to the assessment of these reasons.

3 The antirealist justification of substructural logic

Let us scrutinize more precisely the justification given by radical antirealists for dropping some of the structural rules. The way we proceed will follow our reconstruction of moderate antirealism: we will first consider high-level revisionism, and then low-level revisionism.

3.1 High-level revisionism

Radical antirealism ensues from a strengthening of the epistemic constraints: what does high-level revisionism amount to in this context? A striking feature of the version of radical antirealism we are discussing is that its denial of moderate antirealism's idealizations leads to the rejection of *new* logical laws (the structural rules, instead of just LEM). Something here needs to be explained. Requiring decidability in practice makes the class of problematic sentences larger, but why should such a shift have revisionist implications of a *different* kind?

Our point is the following one. As we have stressed in the first section, the disagreement between realists and moderate antirealists concerns classes of undecidable sentences: the moderate antirealist rejects the disputed logical principle, namely LEM, precisely for those classes. Let us assume that there is an argument Π which relies on the principle that truth should be epistemically constrained and which does show that, for undecidable classes, LEM does not hold (Π is the kind of argument that we have mentioned in section 2). Let us assume furthermore that, in the previous principle, decidability in principle should be replaced by decidability in practice. As a consequence, Π is likely to be turned into a stronger argument Π', which shows that, for domains which are undecidable in practice, LEM does not hold. What is crucial here is that the shift from Π to Π' does not change the logical law (*i.e.* LEM) that is under dispute, but changes the scope of the domain of validity of that law. Arguably, the domain of validity of LEM becomes more restricted: the law is no longer valid for every domain which is decidable in principle, but only for domains which are decidable in practice. To put it another way, those domains which are decidable in principle but not in practice fall outside of the scope of the law.

The point has actually been made by C. Wright in his book on strict finitism:

> "...whereas the intuitionist is content to regard as determinately true or false any arithmetical statement whose truth value can be effectively computed, at least "in principle", the strict finitist will insist that the principle of Bivalence is acceptable only for statement the verification or falsification of which can be guaranteed to be humanly feasible." (in (Wright 1993), p.108)

Along this line, the landscape to be drawn would not be:

realism	$\Rightarrow$	classical logic
moderate antirealism	$\Rightarrow$	intuitionnistic logic
radical antirealism	$\Rightarrow$	linear logic

but rather:

realism	$\Rightarrow$	LEM for all domains
moderate antirealism	$\Rightarrow$	LEM restricted to domains decidable in principle
radical antirealism	$\Rightarrow$	LEM restricted to domains decidable in practice

Restricting LEM to decidable domains and choosing intuitionistic logic is perfectly coherent: the idea is that LEM is not valid in full generality – so that one should choose a logic such as intuitionistic in which the principle is not a theorem. It just happens that for some special domains, the decidable ones, LEM can be used, because of the property these domains have. The same is not true with restricting LEM to a subclass of decidable domains and choosing linear logic: the shift from intuitionistic logic to linear logic cannot be analyzed as a consequence of further restricting the validity of LEM.

To sum up, it is clear that, from a high-level perspective, radical antirealism is bound to yield an even more radical revision. Nonetheless, it is not yet clear why the nature of this revision should be any different from the one advocated by the moderate antirealist. Therefore, if the radical antirealist is to propose a new kind of logic, like linear logic, this justification has to take place from a low-level perspective. And, in any case, the convergence between high-level and low-level that was a nice feature of moderate antirealism will be lost.

3.2 Low-level revisionism

Now we shall turn to low-level revisionism. The question is: can radical antirealism do for linear logic what moderate antirealism does for intuitionist logic? That is, can moderate antirealism provide both a justification to accept the rules of a substructural logic and reasons to reject stronger systems, in particular reasons to reject structural rules?

To answer this question, some preliminary remarks are in order. First, moderate antirealists do put forward a criterion, the criterion of harmony[12],

[12]See paragraph 1.1 above.

which discriminates between acceptable and unacceptable pairs of logical rules. Radical antirealists have to provide an analogous but more demanding criterion. Second, the radical antirealist and the moderate antirealist do not seem at first sight to talk about the same thing. The moderate antirealist focuses on logical rules *stricto sensu*[13], whereas the radical antirealist targets structural rules. A criterion like the principle of harmony is tailored for logical rules – and furthermore for logical rules in a natural deduction format.

Thus in order to provide a complete justification, the moderate and the radical antirealist have to propose admissibility criteria both for structural and for logical rules. Our aim will be to sketch the ways in which these expectations could be fulfilled. Four criteria are thus needed, as can be seen in the following tabular:

	moderate antirealism	radical antirealism
logical criterion	harmony	?
structural criterion	?	?

Let us consider first the admissibility criteria for logical rules (upper line). Right now, there is only one cell whose content is obvious: the moderate antirealist takes the principle of harmony as a requirement on logical rules. It is clear that whatever is required for the moderate antirealist is also required for the radical antirealist: the radical antirealist's logical admissibility criterion has to be at least as strong as the principle of harmony.

But there is a question concerning the means which are available in order to eliminate the detours. For the moderate antirealist, structural rules are always available, in particular, they are available to get a proof of the lower sequent from the upper sequents. But the radical antirealist rejects the structural rules: therefore, when she requires harmony, she will also require that detours can be eliminated *without resorting to structural rules.*

For example, one could take the following rules for conjunction:

$$\frac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \wedge B}\ \wedge\text{-intro} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ \wedge\text{-elim} \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ \wedge\text{-elim}$$

These rules are harmonious as far as the moderate antirealist is concerned, because from:

[13] To our knowledge, Dummett in particular does not discuss the validity of structural rules at all. One contingent reason might be that he uses systems of natural deduction in which structural rules are built-in rather than presented as genuine rules.

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \wedge B}\ \wedge\text{-intro}}{\Gamma, \Gamma' \vdash A}\ \wedge\text{-elim}$$

one can get:

$$\dfrac{\Gamma \vdash A}{\Gamma, \Gamma' \vdash A}\ \text{Weakening}$$

As a consequence, the radical antirealist has to choose so-called "pure" pairs of introduction and elimination rules for which the detours can be eliminated without structural rules. This is for example the case of the following one:

$$\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A\&B}\ \&\text{-intro} \qquad \dfrac{\Gamma \vdash A\&B}{\Gamma \vdash A}\ \&\text{-elim} \qquad \dfrac{\Gamma \vdash A\&B}{\Gamma \vdash B}\ \&\text{-elim}$$

To sum up, the logical criterion for the radical antirealist is just a strengthened version of the principle of harmony, in which the use of structural rules is banned. One obtains the following picture:

	moderate antirealism	radical antirealism
logical criterion	harmony	strong harmony
structural criterion	?	?

2.(a) Let us consider normative criteria for structural rules. What could the moderate antirealist say? To start with, it is important to note that no analogon of the principle of harmony is at hand. Roughly, harmony is meant to show that "nothing new" is introduced, in so far as harmony implies (in *certain* contexts) conservativity. But structural rules *do* introduce some new proof means: there are things which can be proved with structural rules but which cannot be proved without. For example, if we consider a given atomic basis B in the sense of Prawitz (*i.e.* a set of mono-conclusion sequents containing only atomic sentences), it is in general possible that there is a sequent S which is not in B and which can be proven from B by using Weakening or other structural rules.

This means that one has to provide a full-fledged justification of structural rules, which does not rely on some sort of eliminability arguments. We suggest the following principle:

Preservation of effectivity

A structural rule of the form
$$\frac{\Gamma \vdash A}{\Gamma' \vdash A}$$
is admissible iff,

if there exists an effective means to transform justifications for all sentences of Γ into a justification for A, then there exists an effective means to transform justifications for all sentences of Γ' into a justification for A'

The principle of preservation of effectivity is in the spirit of the BHK interpretation for the logical constants. It is applied here at the meta-level to the consequence relation represented by the turnstile. Because of the close connection between the consequence relation at the meta-level and implication at the object-level, it is no surprise that our principle mirrors the BHK clause for implication. The antirealist demands that proofs provide us with effective justifications, nothing less, but nothing more. Therefore, the principle of preservation of effectivity seems to express both necessary and sufficient conditions for the admissibility of structural rules.

This principle validates the standard structural rules. Weakening is admissible:

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A}$$

If one has an effective method to get a justification for A from justifications for sentences in Γ, one has also an effective method to get a justification for A from these justifications plus a justification for B. One just has to *discard* the unnecessary justification for B.

Contraction is admissible as well:

$$\frac{\Gamma, A, A \vdash A}{\Gamma, A \vdash A}$$

If one has an effective method to get a justification for A from justifications for sentences in Γ plus a justification for A and another one for the same sentence A, one has also an effective method to get a justification for A from the justifications for sentences in Γ and the remaining justification for A. One just has to *duplicate* the remaining justification for A whenever needed. (It is crucial here that the effective method provided for the upper sequent has to work whatever justifications for A are given.)

Exchange is admissible as well:

$$\frac{\Gamma, A, B, \Gamma' \vdash A}{\Gamma, B, A, \Gamma' \vdash A}$$

If one has an effective method to get a justification for A from justifications for sentences in Γ, for A, for B and for sentences in Γ', one has also an effective method to get a justification for A from justifications for sentences in Γ, for B, for A and for sentences in Γ'. One just has to look for the required justifications in the right place: the order on the left hand side of the sequent does not matter.

By contrast, let us have a quick look at the following rule, which we might call *stronk*:

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B}\ \textit{stronk}$$

stronk is some kind of strengthening which exhibits the same misbehavior as *tonk*. Let us assume that we have an effective procedure to get a justification for B from justifications for sentences in Γ and a justification for A. It might be the case that such a procedure makes an essential use of the justification provided for A, and that there is no effective procedure giving us a justification for B on the basis of Γ alone. Hence, *stronk* is not an admissible rule.

If we are right, the picture is now this one:

	moderate antirealism	radical antirealism
logical criterion	harmony	strong harmony
structural criterion	preservation of effectivity	?

2.(b) Let us turn now to the central part of the discussion: the radical antirealist perspective on structural rules. More precisely, we will discuss in turn three ways of filling the last blank in our tabular:

(i) Token preservation

(ii) Preservation of local feasibility

(iii) Preservation of global feasibility

(i) As noted above, the crucial claim of the radical antirealist is that one should reject *both* the Weakening rule and the Contraction rule. By contrast, the radical antirealist has no quarrel with the exchange rule. We will start by proposing a criterion that is able to rationalize such a position, and then see whether this criterion can be justified from the antirealist perspective. The natural suggestion is to equate the radical antirealist's structural criterion with the following principle:

> **Principle of Token Preservation**: A structural rule of the form
> $$\frac{\Gamma \vdash A}{\Gamma' \vdash A}$$
> is admissible if, for every formula B, the number of token of B is the same in Γ and in Γ'.

It is easy to see that the principle of Token Preservation rules out the Weakening and Contraction rules. On the contrary, the Exchange rule is admissible according to it. Of course, the pathological *stronk* is ruled out as well.

The Principle of Token Preservation seems thus to mirror adequately the radical antirealist attitude towards the different structural rules. Of course, this is not enough: adopting the principle just on the ground that it yields the desired result would be an entirely *ad hoc* move, if there were no justification of it on account of the basic tenets of radical antirealism.

Here is a first attempt at such a justification of the principle. Linear logicians sometimes motivate their logic by providing an informal semantics for their calculus in terms of *resources and resource consumption* (see (Girard 1995)). On this interpretation, types of formulas stand for types of resources and a sequent of the form $\Gamma \vdash A$ expresses the fact that one can get something of type A from resources corresponding to the elements of Γ. In this perspective, the contraction rule becomes problematic because it says, for instance, that if one may get something of type A from two resources of type B, then one may get something of type A with just one resource of type B. But it is not because one can buy a pack of Marlboros with two bucks that one can buy a pack of Marlboros pack with just one buck. However, it seems to us that this "resource interpretation" is conceptually defective from an antirealist perspective. The reason is the following one. The resource interpretation is based on a sort of causal reading of the turnstile (and, correspondingly, of implication) where formulas stand for events or states of affairs (in terms of the previous example, my owning of two bucks can result in my owning of a pack of Marlboros). But the radical antirealist is concerned with epistemic constraints on speakers. It is fallacious to assimilate the two perspectives. Of course, inference steps have a cognitive cost, and it might well be the case that some inference steps have a cognitive cost significantly higher than some others, so that a radical antirealist should be particularly reluctant to admit them in his favorite logic. But, nonetheless, cognitive resources are not on a par with consumption goods. A justification does not disappear when I use it to build another justification in the same way that buying a pack of cigarettes makes two dollars disappear out of my pocket. For this reason, we do not think that a rejection of structural rules can be based on the "ressource interpretation" of linear logic.

(ii) As a consequence, the radical antirealist has to look for another kind of structural criterion of admissibility. Token preservation does not speak for itself in the epistemic realm; a more principled requirement needs to be provided which would explain why token preservation matters also for justifications after all. The most promising line of thought consists in radicalizing what we have called above the Principle of Preservation of Effectivity. The basic criticism that the radical antirealist addresses to his moderate cousin is that one should require not only effectivity in principle but effectivity in practice or feasibility. Therefore, the natural suggestion is to consider the following principle, which draws on the requirement of feasibility by requiring that rules preserve feasibility of the task consisting in providing justifications:

> **Preservation of Local Feasibility**
>
> A structural rule of the form
>
> $$\frac{\Gamma \vdash A}{\Gamma' \vdash A}$$
>
> is admissible iff,
>
> if there exists a feasible means to transform justifications for all sentences of Γ into a justification for A, there exists a feasible means to transform justifications for all sentences of Γ' into a justification for A

If one accepts to consider this requirement as a consequence of the radical antirealist's position, the crucial question is: does the Preservation of Local Feasibility gives us a reason to reject the Weakening and Contraction rules?

Consider Weakening. Let us assume that we have an effective and *feasible* method to get a justification for A from justifications for sentences in Γ. What kind of method do we get for getting a justification for A from these justifications plus a justification for B? Well, the same as before, except that one has now to drop the justification for A. Why on earth would this not be feasible? After all, the idea is just to do as if no new justification would have been given, and to stick to the good old feasible procedure, *even if* we could now try to use in non feasible ways the new justification we have just been be provided with. But note that for the principle of Preservation of Local Feasiblity to hold, it is sufficient that we have a feasible procedure, because it is clear that there might always be non feasible ways of doing feasible things (try to unload a truck full of sand with a pitchfork instead of a spade).

The same goes with Contraction. Let us assume that we have an effective method to get a justification for A from justifications for sentences in Γ plus a justification for B and another one for the same sentence B. As we said, to get a transformation procedure corresponding to the lower sequent, one has to be able to "re-use" the justification provided for A. Again, why on earth would this not be feasible? By assumption, the justification for A has to be "simple" enough to be dealt with in the transformation procedure. But if this is the case, why would it suddenly cease to be "simple" enough to be reused?

(iii) The radical antirealist might reply that it is intuitively clear that it is harder to get a proof of A from Γ and B than from Γ alone, because one has to take into consideration that B might be necessary to prove A from Γ. If we do not grant this point, the antirealist might claim that our disagreement reflects a brute conflict of intuitions and that our arguments do not bear upon his analysis of what the logic of feasible proofs is. But it is crucial to note that, by saying so, the radical antirealist makes a shift from the question of the preservation of feasibility for transformation procedures to the question of the feasibility of establishing that, say, A follows from Γ. In other words, we now deal with the complexity of the consequence relation itself, *i.e.* with a property of the whole logic, and the admissibility of structural rules should be judged on the basis of their effect on logical systems. This corresponds to a global requirement of feasibility, which bears upon the calculus as a whole, as opposed to the local requirement that we had introduced. This new principle could be spelt in the following way:

> **Preservation of Global Feasibility**
>
> A set of structural rules S preserves global feasibility w.r.t. to a set of logical rules L, iff, if $\vdash_L$ is feasible, then $\vdash_{L+S}$ is feasible as well.

For the radical antirealist's intuitions to be mathematically vindicated, the complexity of the consequence relation of a logic without structural rules should be lower than the complexity of the consequence relation of the same logic to which structural rules have been added. In particular, the radical antirealist seems to be committed to the claim that it is feasible to establish that A follows from Γ in linear logic whereas it is not the case in intuitionistic logic.

However, at this point some well-known results in computability theory and complexity theory precludes such vindication. As a matter of fact, the consequence relation is decidable and PSPACE-complete in intuitionistic logic but is undecidable in full linear logic. If one drops the exponentials, linear logic becomes decidable but is still PSPACE-complete. Furthermore, if one shifts from full linear logic to affine logic, i.e. linear logic plus the weakening rule, one goes from undecidability to decidability (see (Lincoln 1995)). As a consequence, it does not seem that there is a significant correlation between the feasibility of establishing that A is a consequence of Γ and the rejection of structural rules. On a contrary, such a rejection sometimes makes matters worse (see also (Vidal-Rosset 2007), who makes a similar point against strict anti-realism).

4 A way out for radical antirealism?

Up to now, our analysis of radical antirealism has led to two main claims:

- Radical antirealism has to provide a low-level justification of his choice of linear logic.
- Such a justification is still to be provided. In particular, three direct attempts have been examined and shown to fail.

Our criticism of the three putative requirements on structural rules (token preservation, preservation of local feasibility and preservation of global feasibility) are not on a par. For purely mathematical reasons, the idea of preserving global feasibility seems to us to be misguided. The problem with token preservation is its lack of conceptual support: the antirealist does not explain why justifications should share the properties of consumption goods. In a sense, preservation of local feasibility can be construed as some kind of conceptual support in favor of token preservation. However, our criticism of local feasibility suggests that, on this account, an informal notion of justification is not likely to invalidate structural rules.

To be fair, the radical antirealist could blame our failure to see why structural rules are problematic on our informal analysis of justifications. She could claim that, on her view of what justifications are, two justifications can be (substantially) better than one. Now, of course, such a view has to be spelled out. Some help could come from the proof semantics that have been given for linear logic: as long as such a semantics could be considered to provide a formal counterpart for a reasonable notion of justification, it would provide intuitive counterexamples to the admissibility of structurale rules and to the provability of the sequents that can be derived by using them.

The game semantics which have first been proposed by Blass (Blass 1992) provide a case in point. In this setting, justifications are defined as a winning strategies for a designated player on two players games (the two players are P for Player and O for opponent, where P is the designated player who tries to "verify" the formula). Given games for atomic formulas, each complex formula is associated with a mathematically well-defined game, which is defined by recursion on the syntax. The provability of a sequent $\vdash A$ amounts to the existence of a winning of a strategy for P no matter what the atomic games are. Blass gives an example of an infinite game for an atom G such that O has a winning strategy for $G \otimes G$ though she does not have one for G alone. Intuitively, this accounts for the fact that "G and G" (where "and" is multiplicative conjunction) can be harder for me to justify than "G" alone (my opponent might be able to refute my claim that G and G though she is not able to refute my claim that G).

However, a crucial feature of Blass' semantics is the use of infinite two players game, which is responsible for the failure of determinacy and hence for the differences between G and $G \otimes G$. From antirealist perspective, the meaning of infinite "justificatory debates" is not clear. But there are other way to lose determinacy. In particular, natural antirealist constraint on strategies would consist in feasibility requirements: a justification for a formula G should not be any kind of winning strategy, but a feasible one, where feasibility would be captured in terms of a measure of complexity on strategy (say, we should only consider strategies computable by finite automata of a given size to reflect the cognitive limitations of the agent, see (Neyman 1998) for more on this). In a given finite game, one of the two players has some winning strategies, but all these strategies may well fall outside of the class of the feasible ones. Hence the failure of determinacy.

To put it bluntly, there is on the one hand an available story by the radical antirealist which tells us why we should worry about intuitionistic logic and its antirealist foundation and there are on the other hand various available semantics for linear logic which shows for what kind of notions of justifications structural rules can fail to be admissible. A thorough vindication of linear logic by radical antirealism would have to make this story and one of these semantics meet. Our point in the previous section was to suggest that this is by no way an easy task, and that an elaborate notion of justification is needed. Our suggestion in the present section is that game semantics together with complexity constraints on available strategies could be a reasonable candidate.[14]

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[14] In discussion, Greg Restall has suggested another possible line of defense for strict antirealism. By the well-known Curry-Howard correspondence between proofs and programs, proofs in intuitionistic logic can be turned into computable functions represented by λ-terms. Dropping structural rules shrinks the class of typable λ-terms (Weakening allows for empty binding and Contraction for multiple binding): the strict anti-realist would have a point if he could show that the class of functions typable in linear logic corresponds to a more feasible class of functions than those typable in intuitionistic logic.

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