Logical omniscience : from epistemic logic to rational choice

Mikaël Cozic

mikael.cozic@paris4.sorbonne.fr

(Philosophy, Paris IV-Sorbonne and IHPST (CNRS-Paris I))

introduction

- logical omniscience = closure under logical consequence
  $\hookrightarrow$ no deductive ignorance : if someone knows chess rules, he or she knows if White have a winning strategy ; if someone knows Peano axioms, he or she know all theorems of number theory...
- huge idealization :
  "*It is not only mathematicians who need to worry about their failure to know all the consequences of their knowledge. Any context where an agent engages in reasoning is a context that is distorted by the assumption of deductive omniscience, since reasoning (at least deductive reasoning) is an actitivity that deductively omniscient agents have no use for.*" (Stalnaker)

introduction, *cont.*

Plan of the talk :

- epistemic logic : Hintikka, Stalnaker, Fagin, Halpern...
- Question : how extend to probability and decision making ?
- Plan of the talk
  - Section 1 : logical omniscience and epistemic logic
  - Section 2 : the probabilistic case
  - Section 3 : the decision-theoretic case

sec. 1, epistemic propositional language

Definition : The set of **formulas** of an epistemic propositional language $\mathcal{L}B(At)$, $Form(\mathcal{L}B(At))$, is the subset of $\mathcal{L}B(At)$ such that

(i) if $p \in At$, $p \in Form(\mathcal{L}B(At))$,

(ii) if $\phi \in Form(\mathcal{L}B(At))$, then $\neg\phi \in Form(\mathcal{L}B(At))$,

(iii) if $\phi, \psi \in Form(\mathcal{L}B(At))$, then
$(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \longrightarrow \psi) \in Form(\mathcal{L}B(At))$, and

(iv) if $\varphi \in Form(\mathcal{L}B(At))$, then $B\varphi \in Form(\mathcal{L}B(At))$.

(v) only strings of symbols generated by (i)-(iv) in a finite number of steps are in $Form(\mathcal{L}B(At))$.

sec. 1, kripke structures

Definition : let $\mathcal{L}B(At)$ an epistemic propositional language ; a **Kripke structure** for $\mathcal{L}B(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, R)$ where

(i) S is a state space,

(ii) $\pi : At \times S \longrightarrow \{0, 1\}$ is a valuation

(iii) $R \subseteq S \times S$ is an accessibility relation

sec. 1, kripke structures, *cont.*

Definition : $\bar{\pi}$, called the **satisfaction relation**, extends π to every formula of the langage according to the following conditions :

(i) $\bar{\pi}(s, p) = \pi(s, p)$ if $p \in At$

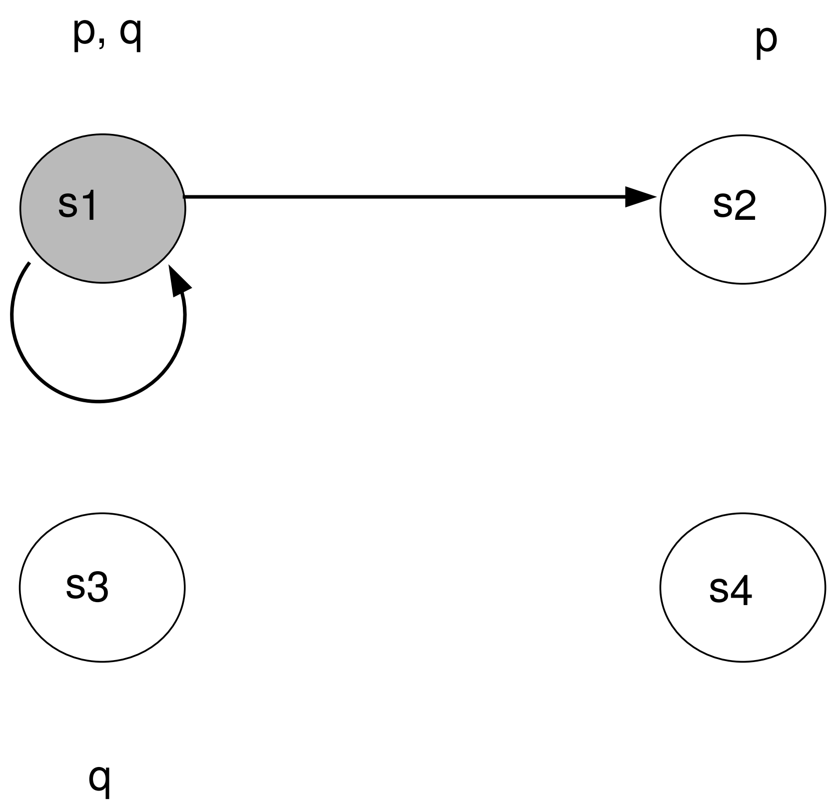
(ii) $\bar{\pi}(s, \phi \wedge \psi) = 1$ iff $\bar{\pi}(s, \phi) = 1$ and $\bar{\pi}(s, \psi) = 1$

(iii) $\bar{\pi}(s, \phi \vee \psi) = 1$ iff $\bar{\pi}(s, \phi) = 1$ or $\bar{\pi}(s, \psi) = 1$

(iv) $\bar{\pi}(s, \neg\phi) = 1$ iff $\bar{\pi}(s, \phi) = 0$

(v) $\bar{\pi}(s, B\phi) = 1$ iff $\forall s'$ s.t. sRs', $\bar{\pi}(s, \phi) = 1$ (***possible-state analysis of belief***)

sec.1, kripke structures, example



sec. 1, logical omniscience

Definitions :

(i) the **proposition** expressed by ϕ, or **informational content** of ϕ is $[[\phi]]_{\mathcal{M}} = \{s : \bar{\pi}(\phi, s) = 1\}$

(ii) ϕ $\mathcal{M}$**-implies** ψ if $[[\phi]]_{\mathcal{M}} \subseteq [[\psi]]_{\mathcal{M}}$

(iii) ϕ and ψ are $\mathcal{M}$**-equivalent** if $[[\phi]]_{\mathcal{M}} = [[\psi]]_{\mathcal{M}}$

Proposition: for all Kripke structure $\mathcal{M}$,

- **Deductive monotony** : if ϕ $\mathcal{M}$-implies ψ, then $B\phi$ $\mathcal{M}$-implies $B\psi$

- **Intensionality** : if ϕ and ψ are $\mathcal{M}$-equivalent, then $B\phi$ and $B\psi$ are $\mathcal{M}$-equivalent

sec.1, neighborhood structures

Definition : a **neighborhood structure** is a 3-tuple $\mathcal{M} = (S, \pi, V)$ where

(i) S is a state space,

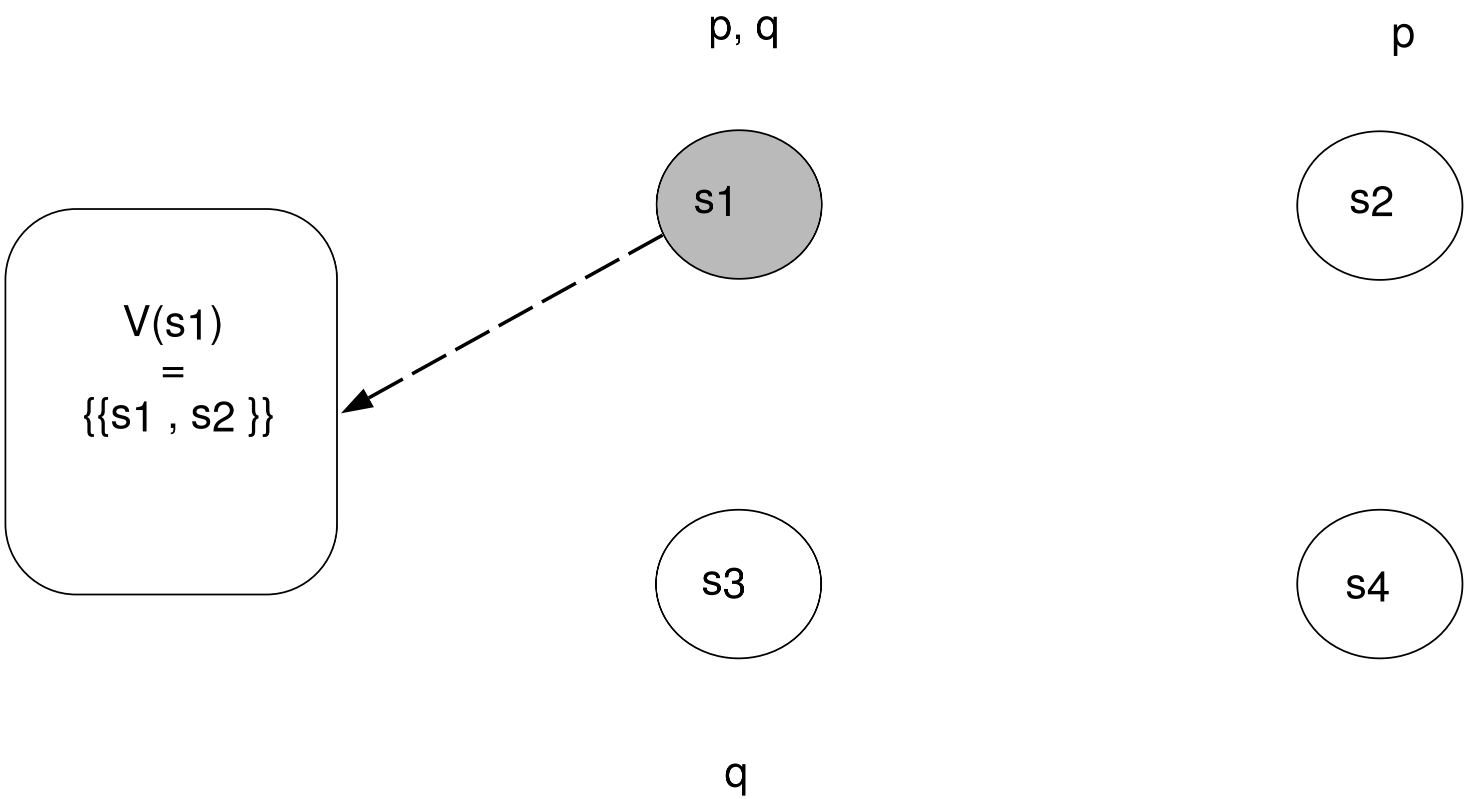
(ii) $\pi : At \times S \longrightarrow \{0, 1\}$ is a valuation,

(iii) $V : S \longrightarrow \wp(\wp(S))$, called the agent's **neighborhood system**, associates to every state a set of propositions.

Doxastic satisfaction condition :

$\bar{\pi}(B\phi, s) = 1$ iff $[[\phi]]_{\mathcal{M}} \in V(s)$

sec.1, neighborhood structures, example



sec.1, awareness structures

Definition : an **awareness structure** is a 4-tuple (S, π, R, A) where

(i) S is a state space,

(ii) $\pi : At \times S \rightarrow \{0, 1\}$ is a valuation,

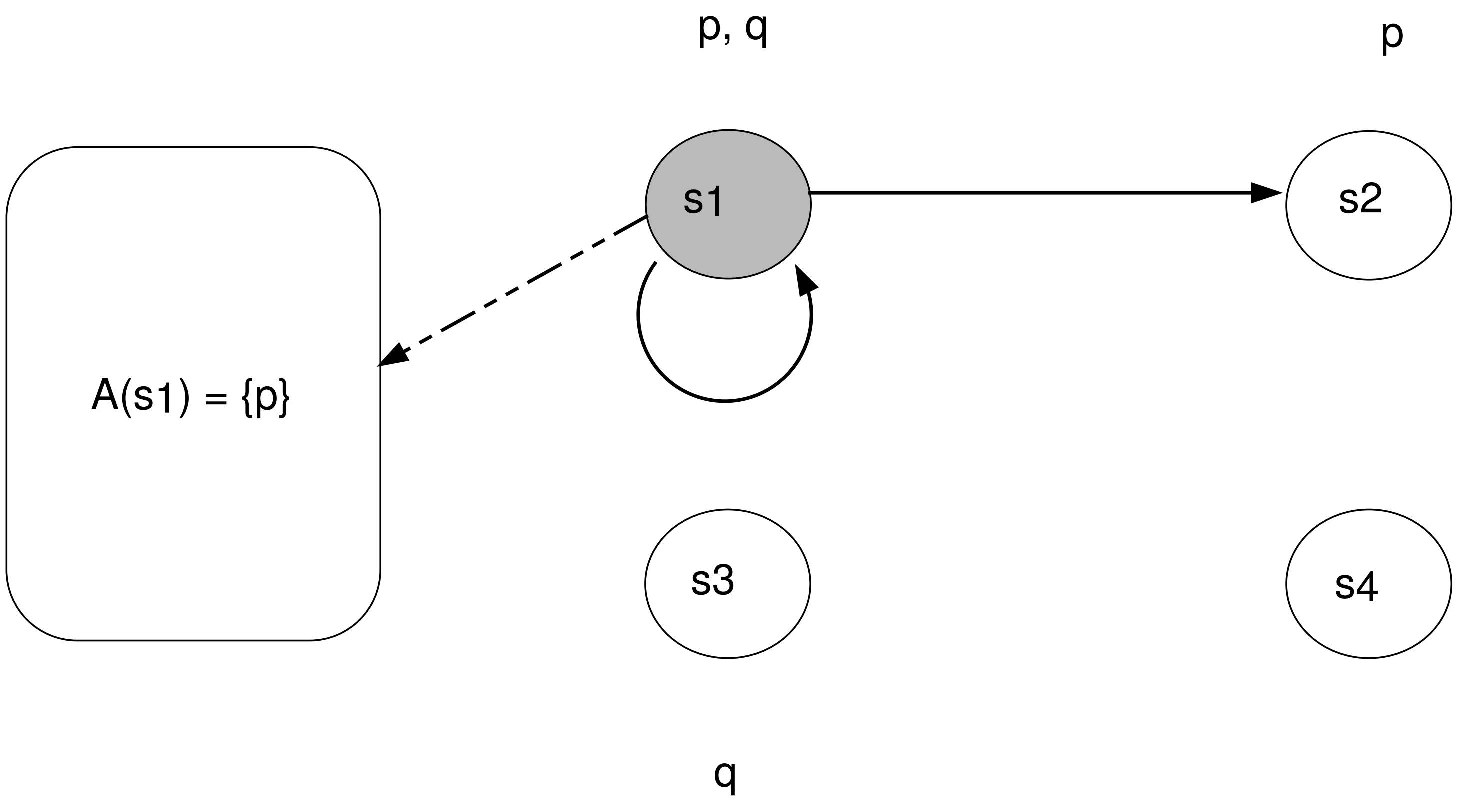
(iii) $R \subseteq S \times S$ is an accessibility relation,

(iv) $A : S \rightarrow Form(\mathcal{L}B(At))$ is a function which maps every state in a set of formulas ("awareness set").

Doxastic satisfaction condition :

$\bar{\pi}(B\phi, s) = 1$ iff $\forall s'$ s.t. sRs', $s' \in [[\phi]]_{\mathcal{M}}$ and $\phi \in A(s)$

sec.1, awareness structures, example



sec.1, non-standard structures

Definition : a **non-standard structure** is a 4-tuple $\mathcal{M} = (S, S', \bar{\pi}, R)$ where

(i) S is a space of standard states,

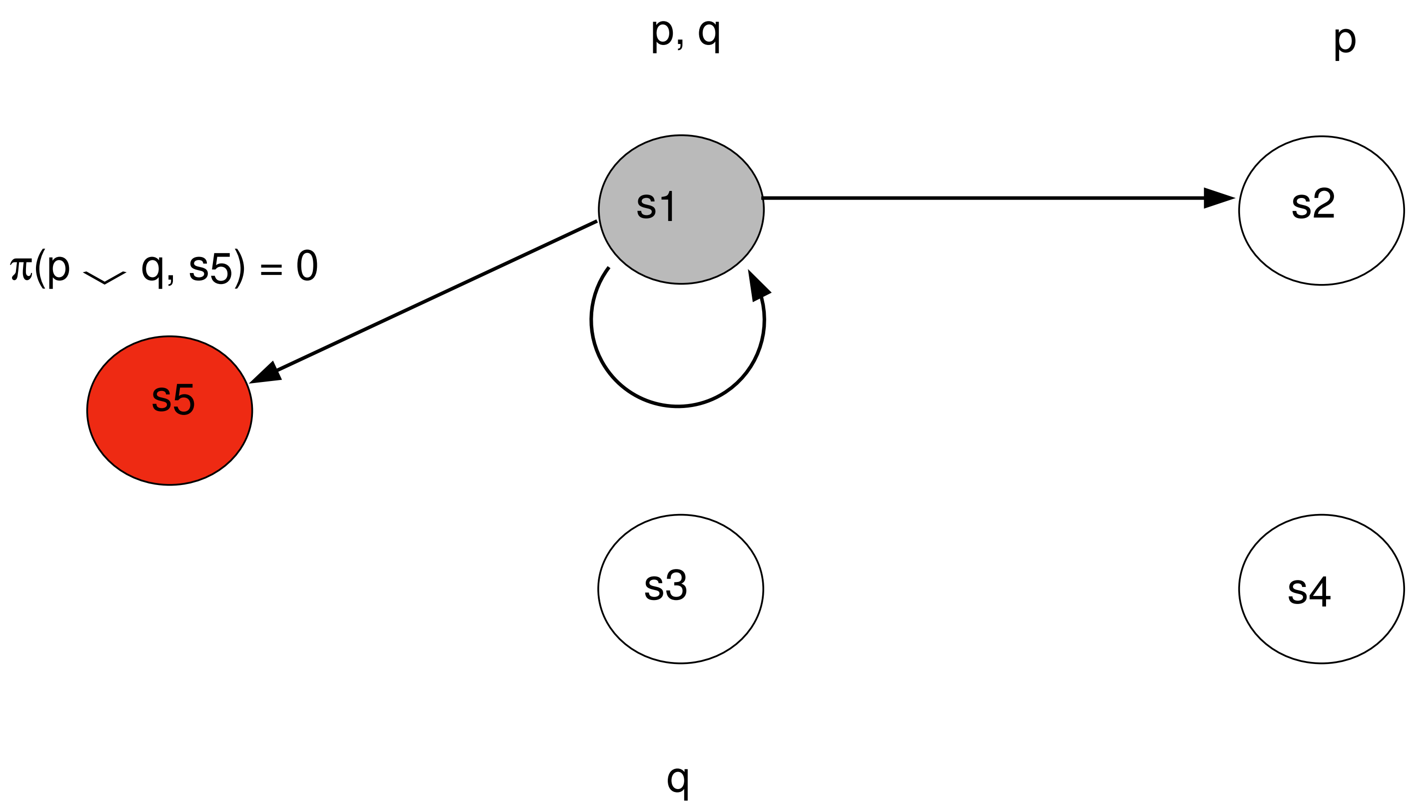
(ii) S' is a space of non-standard states,

(iii) $R \subseteq S \cup S' \times S \cup S'$ is an accessibility relation,

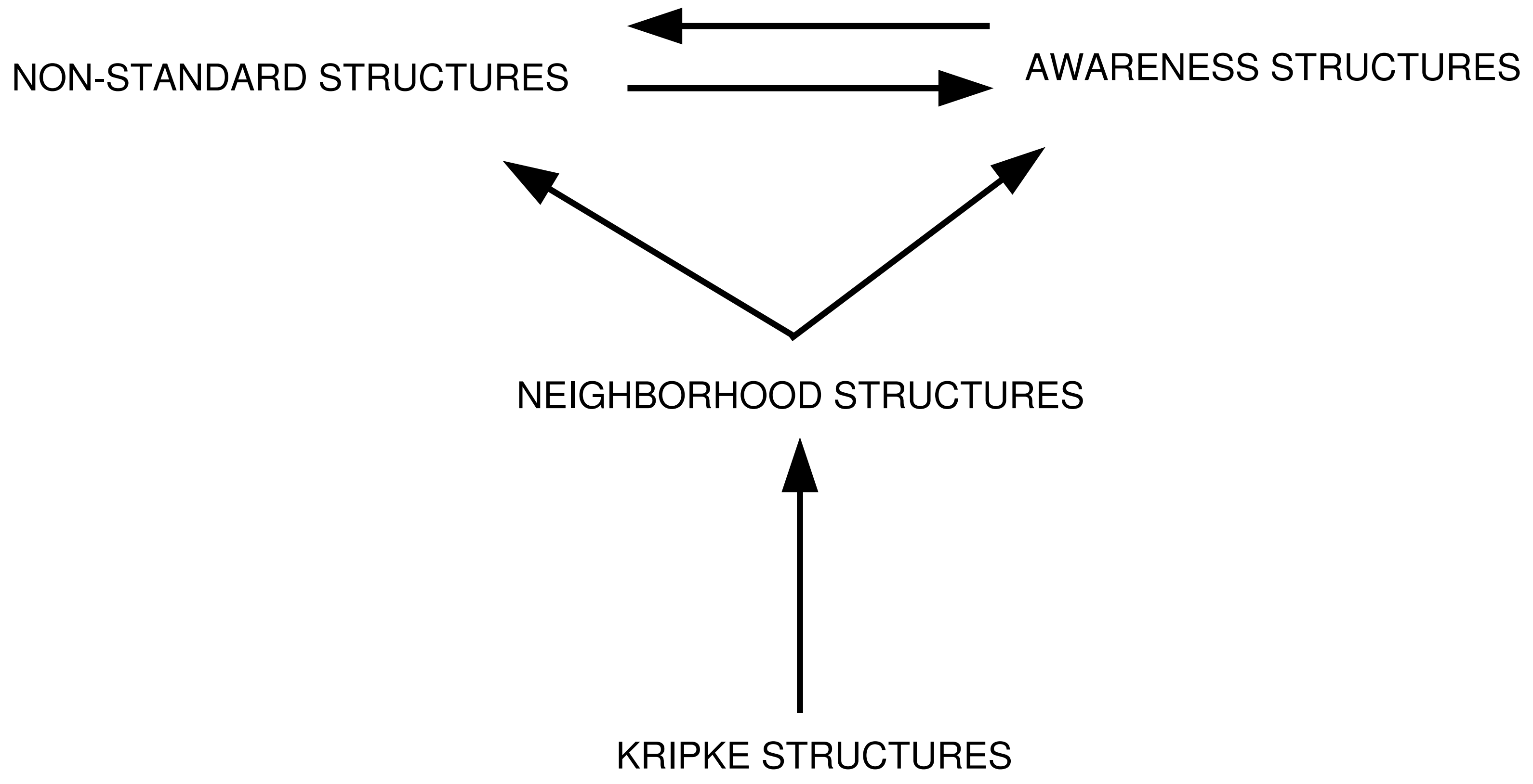
(iv) $\pi : Form(\mathcal{L}B(At)) \times S \rightarrow \{0, 1\}$ is a satisfaction relation **standard on** S

Subjective informational content :
$[[\phi]]^*_{\mathcal{M}} = \{s \in S^* : \pi(\phi, s) = 1\}$

sec.1, non-standard structures, example



sec.1, respective powers



sec. 2, probabilistic structures

Definition : let $\mathcal{L}(At)$ a propositional language ; a **probabilistic structure** for $\mathcal{L}(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, P)$ where

(i) S is a state space,

(ii) π is a valuation,

(iii) P is a probability distribution on S.

An agent believes to degree r a formula $\varphi \in Form(\mathcal{L}(At))$, $PB(\varphi) = r$, if $P([[\varphi]]_{\mathcal{M}}) = r$

sec.2, probabilistic omniscience

Proposition : for all probabilistic structure $\mathcal{M}$,

- **deductive monotony** : if ϕ $\mathcal{M}$-implies ψ, then $PB(\phi) \geq PB(\psi)$.

- **intensionality** : if ϕ and ψ are $\mathcal{M}$-equivalent, then $PB(\phi) = PB(\psi)$.

Case r= 1 :

- if ϕ $\mathcal{M}$-implies ψ, if it is certain that ϕ, it is certain that ψ

- if ϕ and ψ are $\mathcal{M}$-equivalent, ϕ is certain iff ψ is certain.

sec.2, probabilistic non-standard structures

Definition : let $\mathcal{L}(At)$ a propositional language ; a **non-standard probabilistic structure** for $\mathcal{L}(At)$ is a 4-tuple $\mathcal{M} = (S, S', \pi, P)$ where

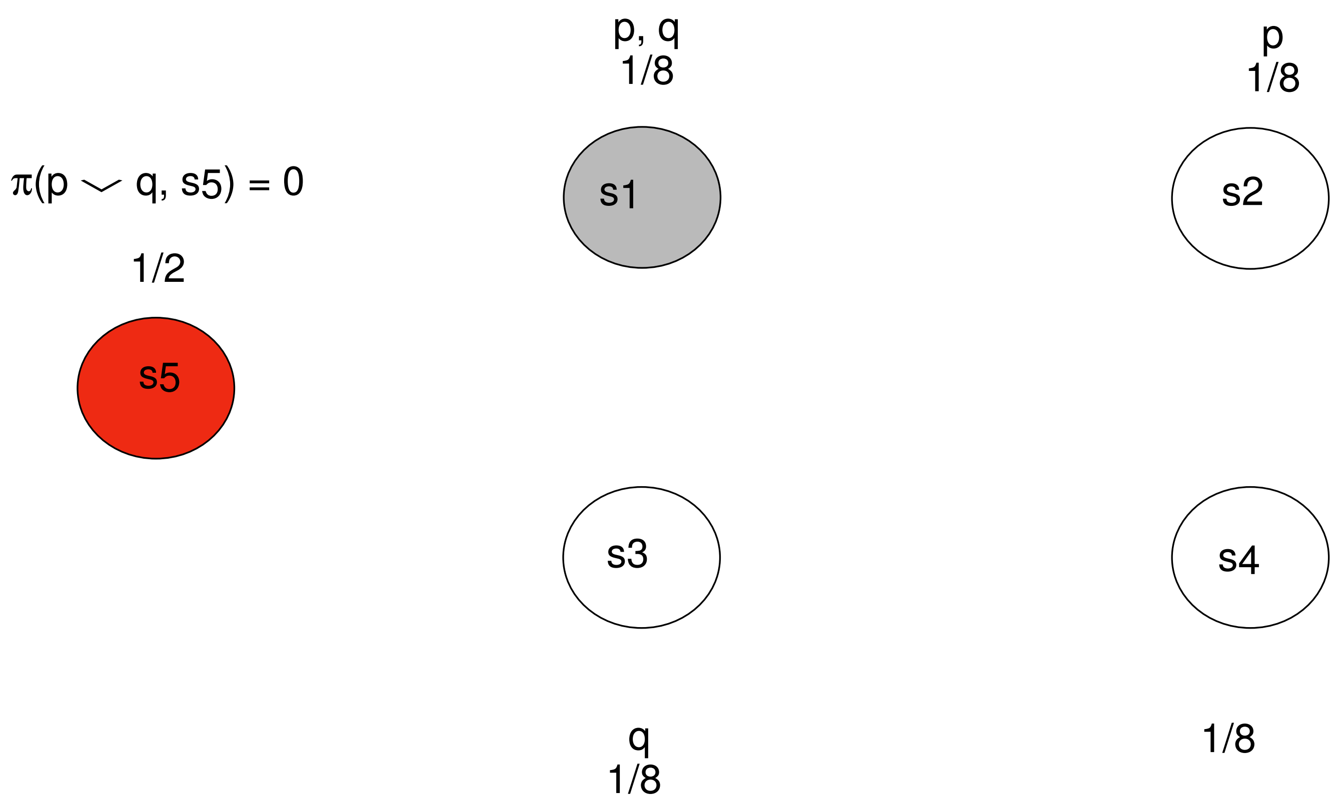
(i) S is a standard state space,

(ii) S' is a non-standard space,

(iii) $\pi : Form(L(At)) \times S \cup S' \longrightarrow \{0, 1\}$ is a satisfaction relation which is standard on S,

(iv) P is a probability distribution on $S^* = S \cup S'$.

sec.2, example



sec.2, special topic, deductive information and learning

- What does it mean to learn that ϕ implies ψ ?
  According to the possible-state analysis of belief : exclude states where ϕ is true but ψ false ; hence, to learn the event

$$I(\phi, \psi) = S^* - ([[\phi]]^*_{\mathcal{M}} - [[\psi]]^*_{\mathcal{M}})$$

- Compatibility between conditionalization on I and deductive monotony :
  Proposition : if $I(\phi, \psi)$ is learned according to Bayes rule, then deductive monotony is regained, *ie* $PB_I(\phi) \leq PB_I(\psi)$.

sec.2, special topic, additivity

- Basic idea : control additivity by means of logical connectives

Definitions :
(i) $\mathcal{M}$ is (logically) **additive** if, when ϕ and ψ are logically incompatible,
$PB(\phi) + PB(\psi) = PB(\phi \vee \psi)$.

(ii) $\mathcal{M}$ is $\vee$**-standard** if for every formulas ϕ, ψ,
$[[\phi \vee \psi]]^*_{\mathcal{M}} = [[\phi]]^*_{\mathcal{M}} \cup [[\psi]]^*_{\mathcal{M}}$.

Proposition If $\mathcal{M}$ is $\vee$-standard, then it is (logically) subaddditive.

sec.2, special topic, additivity, *cont.*

Definitions :
(i) $\mathcal{M}$ satisfies the (logical) inclusion-exclusion rule if
$PB(\phi \vee \psi) = PB(\phi) + PB(\psi) - PB(\phi \wedge \psi)$

(ii) $\mathcal{M}$ satisfies (logical) **submodularity** (resp. supermodularity or convexity) if : $PB(\phi \vee \psi) \leq PB(\phi) + PB(\psi) - PB(\phi \wedge \psi)$ (resp. $PB(\phi \vee \psi) \geq PB(\phi) + PB(\psi) - PB(\phi \wedge \psi)$)

Proposition. Suppose that $\mathcal{M}$ is $\vee$-standard ;

- if $\mathcal{M}$ is negatively $\wedge$-standard, then submodularity holds.

- if $\mathcal{M}$ is positively $\wedge$-standard, then supermodularity holds.

sec.3, model of decision making, primitives

Primitives of decision making under set-theoretic uncertainty :

(i) a set A of opportunities,

(ii) a state space S,

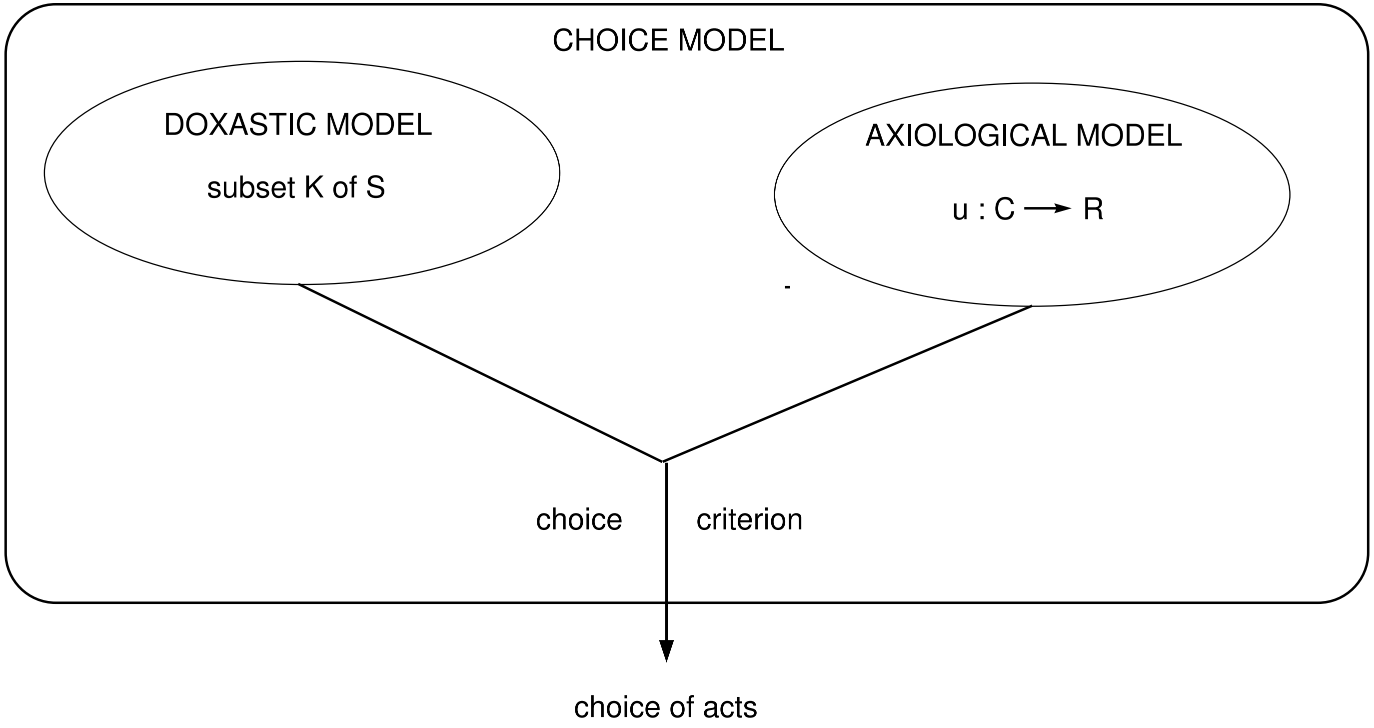
(iii) a set of accessibility $K \subseteq S$,

(iv) a set of consequences C,

(v) a consequence function $\mathcal{C} : A \times S \to C$,

(vi) an utility function $u : C \to \mathbb{R}$.

sec.3, model of decision making, structure



sec. 3, the dilemma

- Question : how to integrate the alternative doxastic model ?

Option 1 (conservative option) : keep the argument's type

NS's case : take as argument the smallest proposition in the neighborhood set

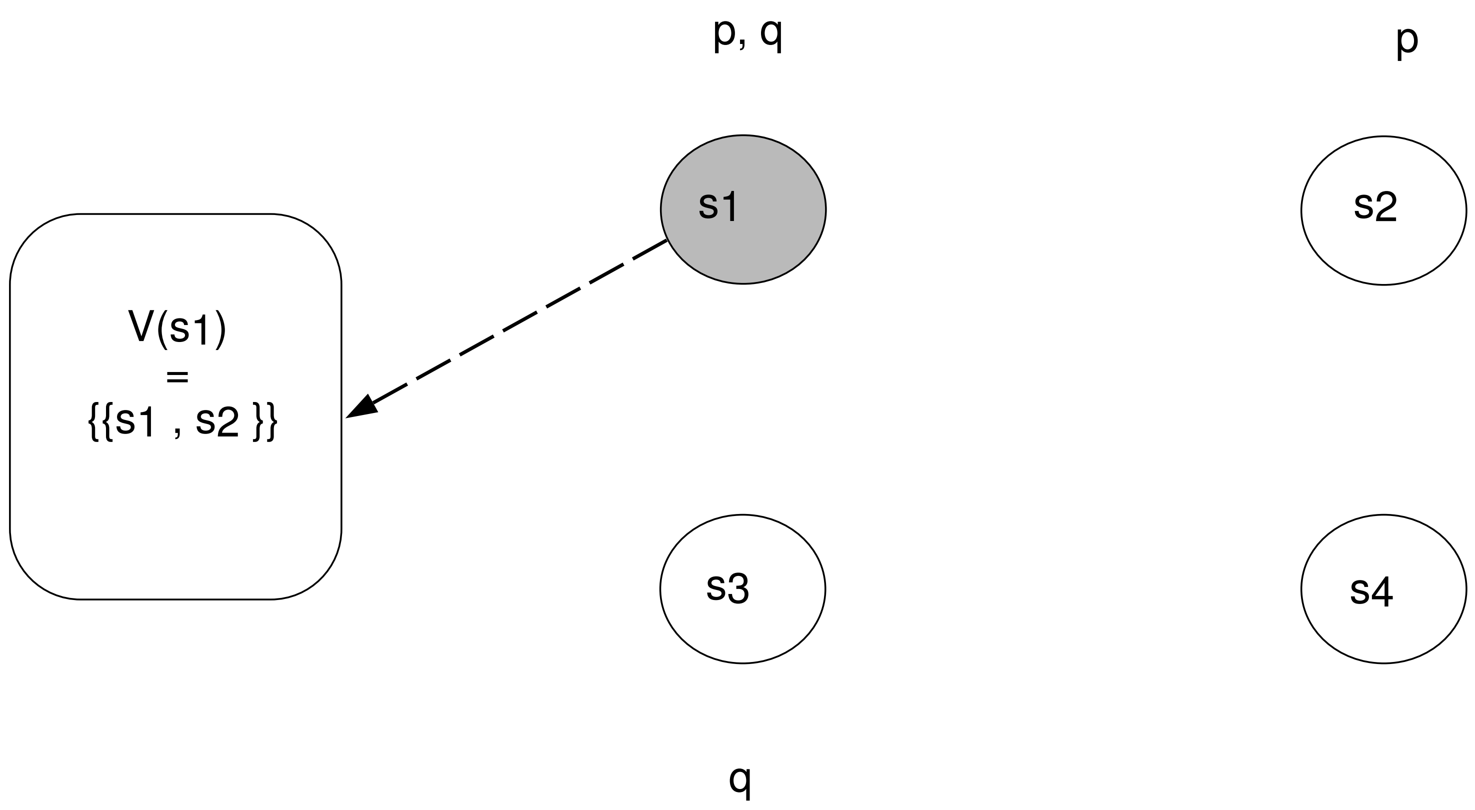
$\hookrightarrow$ makes logical ignorance innocuous

Option 2 (heroic option) : build a new argument's type

NS's case : the whole set of propositions as argument

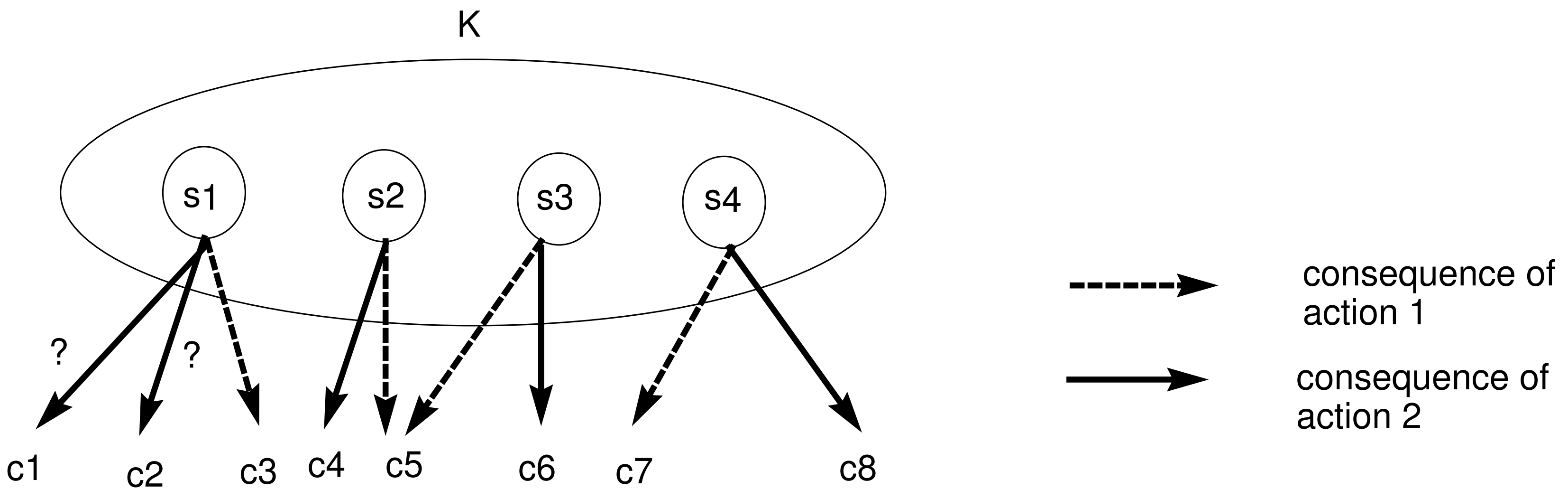
$\hookrightarrow$ if possible, looses naturalness of choice criterion

sec.3, the dilemma, example



sec. 3, non-standard model of choice, target situation

- Target : an agent knows in principle the consequence function, *ie* knows in principle what follows from each pair $(action, state)$, but is not able to infer from that the exact value of each argument.

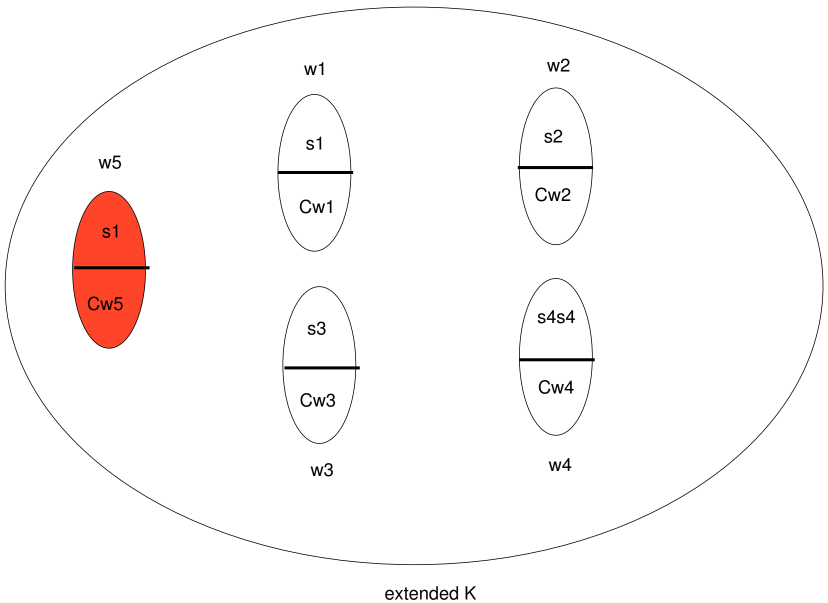


sec. 3, non-standard model of choice, extended states

Definitions :

(i) an **extended state** is a pair $w = (s, \mathcal{C}_w)$ where $s \in S$ and $\mathcal{C}_w : A \to C$

(ii) an extended state $w = (s, \mathcal{C}_w)$ is **standard** if for all $a \in A$, $\mathcal{C}_w(a) = \mathcal{C}^T(a, s)$.



sec.3, applications

- *set-theoretic uncertainty* : consequences on choices according to the criterion
  example : non-standard maximin
  $Sol_{MaxMinNS} = \arg\max_{a \in A} \min_{w \in R} u(\mathcal{C}_w(a))$
  Remark : in set-theoretic uncertainty, for usual choice criterion, non-standard models are equivalent to *consequence correspondences* (Ghirardato 2001) : for any correspondence model, there exists a non-standard equivalent model and conversely

applications, *cont.*

- *expected utility*
  See Lipman 1999
  Classical case : (i) given S and C, (ii) if certain conditions on the preference relation $\preceq$ on $F = C^S$ hold, (iii) then there exists a probability distribution P on S and an utility function u on C s.t. $\preceq$ is SEU-representable.
  Non-standard case : (i) given S and C, (ii') to which conditions on $\preceq$ (iii') does there exist *a non-standard state space* $S^* \supseteq S$, a probability distribution P^* on S^* and an utility function s.t. $\preceq$ is SEU-representable ?

conclusion

- Two main limitations :

  - a framework, not a theory

  - difficulty of choice criterion is omitted