Models of Beliefs for Boundedly Rational Agents

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Abstract

In this tutorial, I will deal with the issue of *cognitive idealizations* in mainstream doxastic models. There are two main kinds of cognitive idealization in these models : logical omniscience and full awareness. The first part of the tutorial (Tutorial 1) is devoted to logical omniscience ; the second part (Tutorial 2), is devoted to full awareness. My broad aim is to defend the so-called impossible states (or impossible worlds) approach pioneered by Hintikka (1975) as a unifying way to weaken these cognitive idealizations. I intend this Tutorial to be a sort of *plea* for impossible states.

Keywords : logical omniscience, unawareness, epistemic logic, probabilistic logic, bounded rationality.

1 Tutorial 1 : Logical Omniscience

1.1 Introduction

Let's imagine an agent that could solve any stochastic decision process, whatever the number of periods, states and alternatives may be ; that could find a Nash Equilibrium in any finite game, whatever the number of players and strategies may be ; more generally, that would have a perfect mathematical knowledge and, still more generally, which would all the logical consequences of his or her beliefs. By definition, this agent would be described as *logically omniscient*.

For sure, logical omniscience is an highly unrealistic hypothesis from the psychological point of view. Yet, this is the cognitive situation of agents in the main current doxastic models, *i.e.* models of beliefs. The issue has been raised a long time ago in epistemic logic (Hintikka (1975), see the recent survey in Fagin, Halpern, Moses & Vardi (1995)), which is the classical model of *full beliefs*. In particular, it has been recognized that logical omniscience is one of the most uneliminable cognitive idealizations, because it is an immediate consequence of the core principle of the modelling : the representation of beliefs by a space of possible states.

My ultimate motivation for studying cognitive idealization lies in rational choice theory. What is the relevance for rational choice theory ? A standard decision model has three fundamental building blocks :

1. a model of beliefs, or doxastic model ;
2. a model of desires, or axiological model, and
3. a criterion of choice, which, given beliefs and desires, selects the "appropriate" actions

In choice under uncertainty, the classical model assumes that the doxastic model is a probability distribution on a state space, the axiological model a utility function on a set of consequences and the criterion is the maximization of expected utility. In this case, the doxastic model is a model of *partial beliefs*. But there are choice models which are built on a model of full beliefs: this is the case of models like maximax or minimax (Luce & Raiffa (1985), chap.13) where one assumes that the agent takes into account the subset of possible states that is compatible with his or her beliefs.

The point is that, in both cases, *the choice model inherits the cognitive idealizations of the doxastic model*. Consequently, the choice model is cognitively *at least as unrealistic as* the doxastic model upon which it is based. Indeed, a choice model is strictly more unrealistic than its doxastic model since it assumes furthermore the axiological model and the implementation of the choice criterion. Hence, one of the main sources of cognitive idealization in choice models is the logical omniscience of their doxastic model ; the weakening of logical omniscience in a decision-theoretic context is therefore one of the main way to build more realistic choice models, *i.e.* to achieve bounded rationality.

Surprisingly, whereas there has been extensive work on logical omniscience in epistemic logic, there has been very few attempts to investigate the extension of the putative solutions to the probabilistic representation of beliefs (*probabilistic case*) and to models of decision making (*decision-theoretic case*)[1].

The aim of Tutorial 1 is to make some progress in filling this gap. Our method is the following one: given that a huge number of (putative) solutions to logical omniscience have been proposed in epistemic logic, we won't start from scratch, but we will consider extensions of the main current solutions. Our main claim is that the solution that we will call the "non-standard structures" constitute the best candidate to this extension.

The remainder of Tutorial 1 proceeds as follows. In section 1, the problem of logical omniscience and its most popular solutions are briefly recalled. Then, it shall be argued that, among these solutions, non-standard structures are the best basis for an extension to probabilistic and decision-theoretic cases. Section 2 is devoted to the probabilistic case and states our main result: an axiomatization for non-standard explicit probabilistic structures. In section 3, we discuss the extension to the decision-theoretic case.

1.2 Logical omniscience in epistemic logic

1.2.1 Epistemic logic

Problems and propositions related to logical omniscience are best expressed in a logical framework, usually called "epistemic logic" (see Fagin et al. (1995) for an extensive technical survey and Stalnaker (1991), reprinted in Stalnaker (1998) for an illuminating philosophical discussion), which is nothing but a particular interpretation of modal logic. Here is a brief review of the classical model: Kripke structures.

First, we have to define the *language* of propositional epistemic logic. The only difference with the language of propositional logic is that this language contains a doxastic operator B : $B\phi$ is intended to mean "the agent believes that ϕ ".

Definition 1
*The set of **formulas** of an epistemic propositional language $\mathcal{L}B(At)$ based on a set At of propositional variables $Form(\mathcal{L}B(At))$, is defined by*

$$\phi ::= p|\neg\phi|\phi \vee \psi|B\phi^2$$

The interpretation of the formulas is given by the famous Kripke structures :

Definition 2
*Let $\mathcal{L}B(At)$ an epistemic propositional language ; a **Kripke structure** for $\mathcal{L}B(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, R)$ where*

(i) S is a state space,

(ii) $\pi : At \times S \rightarrow \{0, 1\}$ is a valuation

(iii) $R \subseteq S \times S$ is an accessibility relation

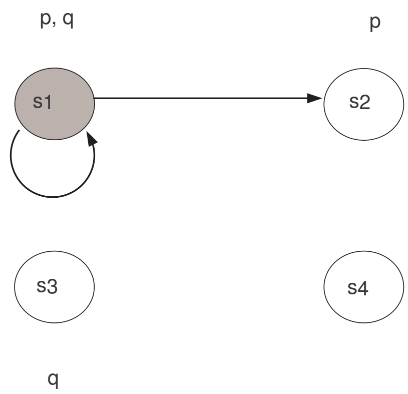

Figure 1: Kripke structures

Intuitively, the accessibility relation associates to every state the states that the agent considers possible given his or her beliefs. π associates to every atomic formula, in every state, a truth value ; it is extended in a canonical way to every formula by the satisfaction relation.

Definition 3
The ***satisfaction relation****, labelled* $\models$*, extends* π *to every formula of the language according to the following conditions :*

(i) $\mathcal{M}, s \models p$ *iff* $\pi(p, s) = 1$

(ii) $\mathcal{M}, s \models \phi \wedge \psi$ *iff* $\mathcal{M}, s \models \phi$ and $\mathcal{M}, s \models \psi$

(iii) $\mathcal{M}, s \models \phi \vee \psi$ *iff* $\mathcal{M}, s \models \phi$ or $\mathcal{M}, s \models \psi$

(iv) $\mathcal{M}, s \models \neg\phi$ *iff* $\mathcal{M}, s \not\models \phi$

(v) $\mathcal{M}, s \models B\phi$ *iff* $\forall s'$ *s.t.* sRs', $\mathcal{M}, s' \models \phi$

The specific doxastic condition contains what might be called the **possible-state analysis of belief**. It means that an agent believes that ϕ if, in all the states that (according to him or her) could be the actual state, ϕ is true : *to believe something is to exclude that it could be false.* Conversely, an agent doesn't believe ϕ if, in some of the states that could be the actual state, ϕ is false : *not to believe is to consider that it could be false.* This principle will be significant in the discussions below.

Example 1
$S = \{s_1, s_2, s_3, s_4\}$ *; p ("it's sunny") is true in* s_1 *and* s_2*, q ("it's windy") in* s_1 *and* s_4*. Suppose that* s_1 *is the actual state and that in this state the agent believes that p is true but does not know if q is true. Figure 1 represents this situation, omitting the accessibility relation in the non-actual states.*

Definition 4
Let $\mathcal{M}$ *be a Kripke structure ; in* $\mathcal{M}$*, the set of states where* ϕ *is true, or the* ***proposition*** *expressed by* ϕ*, or the* ***informational content*** *of* ϕ*, is noted* $[[\varphi]]_{\mathcal{M}} = \{s : \mathcal{M}, s \models \varphi\}$.

To formulate logical omniscience, we need lastly to define the following semantical relations between formulas.

Definition 5
*ϕ **$\mathcal{M}$-implies** ψ if $[[\phi]]_{\mathcal{M}} \subseteq [[\psi]]_{\mathcal{M}}$. ϕ and ψ are **$\mathcal{M}$-equivalent** if $[[\phi]]_{\mathcal{M}} = [[\psi]]_{\mathcal{M}}$*

There are several forms of logical omniscience (see Fagin et al. (1995)) ; the next proposition shows that two of them, deductive monotony and intensionality, hold in Kripke structures :

Proposition 1
Let $\mathcal{M}$ be a Kripke structure and $\phi, \psi \in \mathcal{L}B(At)$;

*(i) **Deductive monotony** : if ϕ $\mathcal{M}$-implies ψ, then $B\phi$ $\mathcal{M}$-implies $B\psi$*

*(ii) **Intensionality** : if ϕ and ψ are $\mathcal{M}$-equivalent, then $B\phi$ and $B\psi$ are $\mathcal{M}$-equivalent*

Both properties are obvious theorems in the axiom system K, which is sound and complete for Kripke structures :

System K
(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ
(K) $B\phi \wedge B(\phi \rightarrow \psi) \rightarrow B\psi$
(RN) From ϕ, infer $B\phi$

1.2.2 Three solutions to logical omniscience

A huge number of solutions have been proposed to weaken logical omniscience, and arguably no consensus has been reached (see Fagin et al. (1995))[3]. We identify three main solutions to logical omniscience, which are our three candidates to an extension to the probabilistic or decision-theoretic case. There is probably some arbitrariness in this selection, but they are among the most used, natural and powerful existing solutions.

1.2.3 Neighborhood structures

The "neighborhood structures", sometimes called "Montague-Scott structures" are our first candidate. The basic idea is to make explicit the *propositions* that the agent believes ; the neighborhood system of an agent at a given state is precisely the set of propositions that the agent believes.

Definition 6
*A **neighborhood structure** is a 3-tuple $\mathcal{M} = (S, \pi, V)$ where*

(i) S is a state space,

(ii) $\pi : At \times S \rightarrow \{0, 1\}$ is a valuation,

*(iii) $V : S \rightarrow \wp(\wp(S))$, called the agent's **neighborhood system**, associates to every state a set of propositions.*

The conditions on the satisfaction relation are the same, except for the doxastic operator :

$$\mathcal{M}, s \vDash B\phi \text{ iff } [[\phi]]_{\mathcal{M}} \in V(s)$$

It's easy to check that deductive monotony is invalidated by neighborhood structures, as shown by the following example.

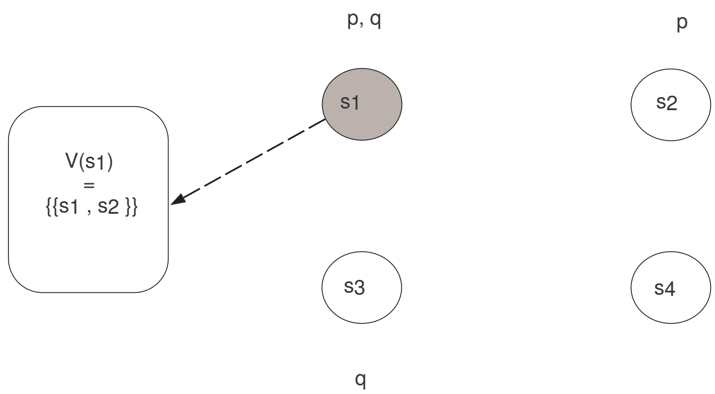

Figure 2: Neighborhood structures

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Example 2
Let's consider the first example and replace the accessibility relation by a neighborhood system ; $V(s_1)$ contains $\{s_1, s_2\}$ but not $\{s_1, s_2, s_3\}$. Then, in s_1, Bp is true but not $B(p \vee q)$. This is represented in Figure 2[4].

As expected, one can regain deductive monotony by closing the neighborhood systems under supersets. Nonetheless, the axiomatization presented below makes clear that the power of neighborhood system is limited: intensionality cannot be weakened.

System E (Chellas, 1980)
(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ
(RE) From $\phi \leftrightarrow \psi$ infer $B\phi \leftrightarrow B\psi$

1.2.4 Awareness© structures

The second solution, due to J. Halpern and R. Fagin (Fagin & Halpern (1988)[5]) , are the "awareness© structures". The basic idea is to put a *syntactical filter* on the agent's beliefs. The term "awareness" suggests that this can be interpreted as reflecting the agent's awareness state (see Tutorial 2), but other interpretations are conceivable as well.

Definition 7
*An **awareness© structure** is a 4-tuple (S, π, R, A) where*

(i) S is a state space,

(ii) $\pi : At \times S \rightarrow \{0, 1\}$ is a valuation,

(iii) $R \subseteq S \times S$ is an accessibility relation,

(iv) $A : S \rightarrow Form(\mathcal{L}B(At))$ is a function which maps every state in a set of formulas ("awareness© set").

The new condition on the satisfaction relation is the following:

$\mathcal{M}, s \models B\phi$ iff $\forall s'$ s.t. sRs', $s' \in [[\phi]]_{\mathcal{M}}$ and $\phi \in A(s)$

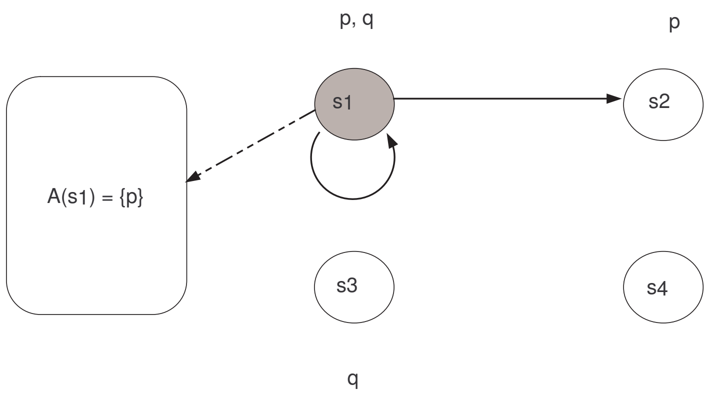

Figure 3: Awareness© structures

This new doxastic condition permits to weaken *any form* of logical omniscience ; in particular, our example shows how to model an agent who violates deductive monotony.

Example 3
Let's consider our example and stipulate that $A(s_1) = \{p\}$. Then it is still the case that Bp is true in s_1, but not $B(p \vee q)$. This is represented in Figure 3.

If one keeps the basic language $\mathcal{L}B(At)$, one obtains as axiom system a minimal epistemic logic which eliminates any form of logical omniscience:

Minimal Epistemic Logic (FHMV 1995)
(PROP) Instances of propositional tautologies (MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

1.2.5 Non-standard structures

We now switch to our last solution : the non-standard structures, which are sometimes called "Kripke structures with impossible states". Contrary to the two preceding solutions, neither the accessibility relation nor the doxastic condition are modified. What is revised is the underlying state space or, more precisely, the nature of the satisfaction relation in certain states of the state space.

Definition 8
*A **non-standard structure** is a 5-tuple $\mathcal{M} = (S, S', \pi, R, \models)$ where*

(i) S is a space of standard states,

(ii) S' is a space of non-standard states,

(iii) $R \subseteq S \cup S' \times S \cup S'$ is an accessibility relation,

(iv) $\pi : Form(\mathcal{L}B(At)) \times S \rightarrow \{0, 1\}$ is a valuation on S

(v) $\models$ is a satisfaction relation which is standard on S (recursively defined as usual) but arbitrary on S'

In non-standard structures, there are no *a priori* constraints on the satisfaction relation in non-standard states. For instance, in a non-standard state s', both ϕ and $\neg\phi$ can be

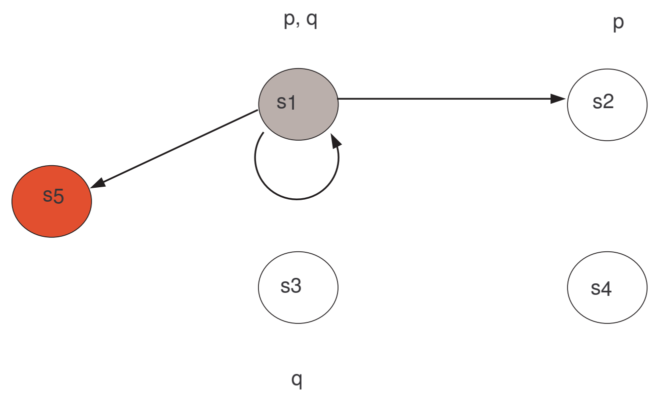

Figure 4: Non-standard structures

false. For every formula ϕ, one might therefore distinguish its *objective informational content* $[[\phi]]_{\mathcal{M}} = \{s \in S : \mathcal{M}, s \vDash \phi\}$ from its *subjective informational content* $[[\phi]]^*_{\mathcal{M}} = \{s \in S^* = S \cup S' : \mathcal{M}, s \vDash \phi\}$. In spite of appearances, this generalization of Kripke structures is arguably natural as soon as one accepts the possible-state analysis of beliefs. Recall that, according to this analysis,

- to believe that ϕ is to exclude that ϕ could be false, and
- not to believe that ψ is not to exclude that ψ could be false.

In consequence, according to the possible-state analysis, to believe that ϕ but not to believe one of its logical consequences ψ is to consider as possible at least one state where ϕ is true but ψ false. By definition, a state of this kind is logically non-standard. Non-standard structures is the most straightforward way to keep the possible-state analysis of beliefs[6].

Example 4
Let's consider our example but add a non-stantard state in $S' = s_5$; we stipulate that $\mathcal{M}, s_5 p$, but that $\mathcal{M}, s_5 \nvDash (p \vee q)$. Then in s_1, Bp is true but not $B(p \vee q)$. This is represented in Figure 4.

1.3 The probabilistic case

Mainstream decision theory is based on doxatic models of *partial beliefs*, not of full beliefs. Hence weakenings of logical omniscience in the framework of epistemic logic does not give directly a way to weaken logical omniscience that is appropriate for decision theorists. The aim of this section is to study the probabilistic extension of doxastic models without logical omniscience.

1.3.1 Probabilistic counterpart of logical omniscience

First, we have to define the probabilistic counterparts of logical omniscience. In the usual (non-logical) framework, if P is a probability distribution on S[7], then the following properties are the counterparts of logical omniscience :

- if $E \subseteq E'$, then $P(E) \leq P(E')$,

- if $E = E'$, then $P(E) = P(E')$.

But to be closer to the preceding section, it is better to work with an elementary[8] logical version of the usual probabilistic model :

Definition 9
Let $\mathcal{L}(At)$ a propositional language ; a ***probabilistic structure***[9] *for $\mathcal{L}(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, P)$ where*

(i) S is a state space,

(iii) π is a valuation,

(iv) P is a probability distribution on S.

We will say that an agent believes to degree r a formula $\phi \in Form(\mathcal{L}(At))$, symbolized by $CP(\phi) = r$, if $P([[\phi]]_{\mathcal{M}}) = r$[10]. We can state the precise probabilistic counterparts of logical omniscience:

Proposition 2
The following holds in probabilistic structures :

(i) deductive monotony : if ϕ $\mathcal{M}$-implies ψ, then $CP(\phi) \leq CP(\psi)$.

(ii) intensionality : if ϕ and ψ are $\mathcal{M}$-equivalent, then $CP(\phi) = CP(\psi)$.

One can check that these are indeed the *counterparts* of logical omniscience by looking at the limit case of *certainty*, *i.e.* of maximal degree of belief : (i) if an agent is certain that ϕ and if ϕ $\mathcal{M}$-implies ψ, then the agent is certain that ψ as well ; (ii) if ϕ and ψ are $\mathcal{M}$-equivalent, then an agent is certain that ϕ iff he or she is certain that ψ.

Which of the three solutions to choose for this extension ?

(a) First, we should eliminate neighbordhood structures because their power is limited: intensionality is a too strong idealization. This is especially sensitive in a decision context, where, under the label of "framing effects", it is has been recognized for a long time that logically equivalent formulations of a decision problem could lead to different behaviors.

(b) Second, the extension of awareness© structures seems intrisically tricky. Suppose that an agent believes ϕ to degree r_ϕ and ψ to degree r_ψ with ϕ $\mathcal{M}$-implying ψ and $r_\phi > r_\psi$. This is a failure of deductive monotony. Now, in an analogous situation, the way awareness© structures proceed in epistemic logic is by "dropping" the formula ψ. Let's apply this method to the probabilistic case: we would say that an agent believes that ϕ to degree r if $P([[\phi]]_{\mathcal{M}}) = r$ *and* he or she is aware of ϕ. But no one could model a situation like the preceding one: either the agent is aware of ψ and in this case necessarily he or she believes that ψ to a degree $r_\psi \geq r_\phi$; or he or she is not aware of ψ, and it this case he or she has no degree of belief toward ψ. This is not a knock-down argument, but it implies that if one wants to extend awareness© structures, one has to sophisticate it substantially.

(c) Lastly, the extension of awareness© structure is problematic in our perspective, *i.e.* a perspective of decision-theoretic application. To see why, let's notice that a criterion choice like expected utility might be seen as a function whose first argument is a doxastic model and second argument an axiological model. If we would extend the awareness© structures, the first value of an expected utility criterion would not be any more a simple probability distribution. Consequently, *we should have to revise our choice criterion*. For sure, nothing precludes such a move, but simplicity recommends another tactic.

We are therefore left with non-standard structures. Non-standard structures do not suffer from the above mentioned troubles : they are as powerful as one can wish, the extension is intrisically simple and they should permit to keep usual choice criterion when embedded in a choice model. This is our motivation, but now we have to turn to positive arguments[11].

1.3.2 Non-standard implicit probabilistic structures

To give the basic insights and show the fruitfulness of the proposition, we will continue to continue to work in the elementary setting where no doxastic operators are in the object-language.

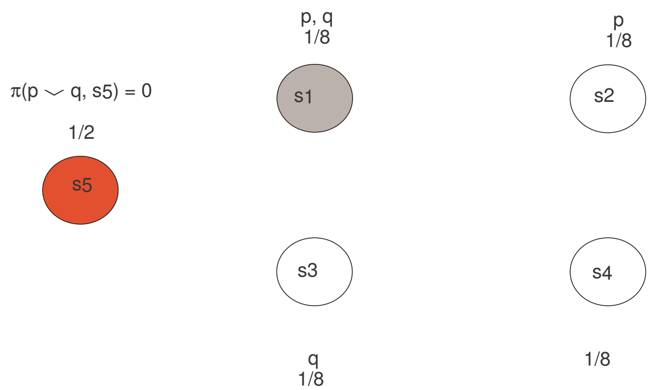

Figure 5: Probabilistic non-standard structures

Definition 10
Let $\mathcal{L}(At)$ a propositional language ; an ***non-standard implicit probabilistic structure*** *for $\mathcal{L}(At)$ is a 5-tuple $\mathcal{M} = (S, S', \pi, \vDash, P)$ where*

(i) S is a standard state space,

(ii) S' is a non-standard space,

(iii) $\pi : Form(L(At)) \times S \rightarrow \{0, 1\}$ is a valuation on S,

(iv) $\vDash$ is a satisfaction relation which is standard on S but arbitrary on S'

(v) P is a probability distribution on $S^* = S \cup S'$.

As in the set-theoretic case, one can distinguish the objective informational content of a formula, *ie* the standard states where this formula is true, and the subjective informational content of a formula, *ie* the states where this formula is true.

To obtain the expected benefit, the non-standard probabilistic structures should characterize the agent's doxastic state on the basis of *subjective* informational content : an agent believes a formula ϕ to degree r, $CP(\phi) = r$, if $P([[\phi]]^*_{\mathcal{M}}) = r$. It is easy to check that, in this case, logical omniscience can be utterly controlled.

Example 5
Let's take the same space state as in the preceding examples. Suppose that the agent has the following partial beliefs : $CP(p) > CP(p \vee q)$. This can be modelled in the following way : $S' = \{s_5\}$, $s_5 \in [[p]]^*_{\mathcal{M}}$ but $s_5 \notin [[p \vee q]]^*_{\mathcal{M}}$, $P(s_1) = P(s_2) = P(s_3) = P(s_4)1/8$ and $P(s_5) = 1/2$. This is represented in Figure 5.

1.3.3 Special topics : deductive information and additivity

This extension of non-standard structures is admittedly straightforward and simple. It gives immediately the means to weaken logical idealizations. Furthermore, it opens perspectives specific to the probabilistic case; two of them will be briefly mentioned.

Deductive information and learning First, one can model the fact that an agent acquire not only empirical information but *deductive information*; in non-standard structures, this corresponds to the fact that *the agent eliminates non-standard states*.

Let's come back to our generic situation. Suppose that our agent learns that ϕ implies ψ. This means that he or she learns that the states where ϕ is true but ψ false are impossible. This is equivalent to say that he or she learns the event

$$I = S^* - ([[\phi]]^*_{\mathcal{M}} - [[\psi]]^*_{\mathcal{M}})$$

To be satisfying, such a notion of deductive information must respect a requirement of compatibility between revising and logical monotony: if the agent learns that ϕ implies ψ and revise his or her beliefs upon this fact, his or her new probability distribution should conform to logical monotony with respect to ϕ and ψ. One can check that it is the case with the main revising rule, *i.e.* conditionalization.

Proposition 3
If I is learned following the conditionalization, then deductive monotony is regained, ie $CP_I(\phi) \leq CP_I(\psi)$.

Example 6
This can be checked in the preceding example : $I = S = \{s_1, s_2, s_3, s_4\}$. By conditionalization, $CP_I(p) = 1/2$ whereas $CP_I(p \vee q) = 3/4$.

Additivity A second topic is additivity. From a logical point of view, one can define additivity as follows :

Definition 11
$\mathcal{M}$ is (logically) ***additive*** *if, when ϕ and ψ are logically incompatible, $CP(\phi) + CP(\psi) = CP(\phi \vee \psi)$.*

Additivity is of course the core of the probabilistic representation of beliefs, and alternative representations of beliefs depart often from probability on this point. For example, in the Dempster-Shafer theory (Shafer (1976)), the so-called belief function is superadditive (in our notation, $CP(\phi \vee \psi) \geq CP(\phi) + CP(\psi)$) whereas its dual, the plausibility function, is subadditive ($CP(\phi \vee \psi) \leq CP(\phi) + CP(\psi)$).

A noteworthy aspect of probabilistic non-standard structures is that the freedom of the connectives' behavior in non-standard states permits us to have a very flexible framework with respect to additivity : simple conditions on the connectives imply general properties concerning additivity.

Definition 12
Let $\mathcal{M} = (S, S', \pi, \vDash, P)$ a non-standard probabilistic structure ; $\mathcal{M}$ is ***$\vee$-standard*** *if for every formulas ϕ, ψ, $[[\phi \vee \psi]]^*_{\mathcal{M}} = [[\phi]]^*_{\mathcal{M}} \cup [[\psi]]^*_{\mathcal{M}}$.*

This means that the disjonction behaves in the usual way in non-standard states ; a trivial consequence of this is that the structure $\mathcal{M}$ is (logically) subadditive.

Proposition 4
If $\mathcal{M}$ is $\vee$-standard, then it is (logically) subaddditive.

To be a little bit more general, one can consider the (logical) inclusion-exclusion rule :

$$CP(\phi \vee \psi) = CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$$

One can define (logical) **submodularity** (resp. supermodularity or convexity) as : $CP(\phi \vee \psi) \leq CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$ (resp. $CP(\phi \vee \psi) \geq CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$).

It's clear that to control submodularity, we have to control the conjunction's behavior.

Definition 13
Let $\mathcal{M} = (S, S', \pi, \vDash, P)$ a probabilistic non-standard structure ;

(i) $\mathcal{M}$ is ***negatively $\wedge$-standard*** *if for every formulas ϕ, ψ, when $\mathcal{M}, s \nvDash \phi$ or $\mathcal{M}, s \nvDash \psi$, then $\mathcal{M}, s \nvDash \phi \wedge \psi$.*

(ii) $\mathcal{M}$ is ***positively $\wedge$-standard*** *if for every formulas ϕ, ψ, when $\mathcal{M}, s \vDash \phi$ and $\mathcal{M}, s \vDash \psi$, then $\mathcal{M}, s \vDash \phi \wedge \psi$.*

Proposition 5
Suppose that $\mathcal{M}$ is $\vee$-standard ;
- if $\mathcal{M}$ is negatively $\wedge$-standard, then submodularity holds.
- if $\mathcal{M}$ is positively $\wedge$-standard, then supermodularity holds.

Proof : see the Appendix.

1.3.4 Non-standard explicit probabilistic structures

Implicit probabilistic structures are not very expressive ; to have a true analogon of epistemic logic, we have to start from an object-language that contains (partial) doxastic operator.

Following R. Aumann (Aumann (1999)) and A. Heifetz and Ph. Mongin (Heifetz & Mongin (2001)), we consider the operator L_a[12][13]. The intuitive meaning of $L_a\phi$ is: the agent believes at least to degree a that ϕ. Note that we add the usual symbols $\top$, $\bot$: $\top$ is what the agent recognizes as necessarily true and $\bot$ is what he or she recognizes as necessarily false.

Definition 14
The set of formulas of an explicit probabilistic language $\mathcal{L}L(At)$ based on a set At of propositional variables, $Form(\mathcal{L}L(At))$ is defined by :

$$\phi ::= p|\bot|\top|\neg\phi|\phi \vee \psi|L_a\phi$$

where $p \in At$ and $a \in [0,1] \subseteq \mathbb{Q}$.

The corresponding structures are an obvious extension of implicit non-standard structures:

Definition 15
*A **non-standard explicit probabilistic structure** for $\mathcal{L}L_a(At)$ is a 5-tuple $\mathcal{M} = (S, S'\pi, \models, P)$ where*

(i) $\models$ is a satisfaction relation s.t.
  (a) $\models$ is standard on S for all propositional connectives
  (b) $\forall s \in S$, $\mathcal{M}, s \models L_a\phi$ iff $P(s)([[\phi]]^*_{\mathcal{M}}) \geq a$
  (c) $\forall s \in S \cup S'$, $\mathcal{M}, s \models \top$ and $\mathcal{M}, s \not\models \bot$

(ii) $P : S^* \to \Delta(S^*)$ assigns to every state a probability distribution on the state space.

In Aumann (1999), R. Aumann has failed to axiomatize (standard) explicit probabilistic structures, but Heifetz & Mongin (2001) have recently devised an axiom system that is (weakly) complete for these structures. In comparison with epistemic logic, one of the problems is that the adaptation of the usual proof method, *i.e.* the method of canonical models, is not trivial. More precisely, in the epistemic logic's case, it is easy to define a canonical accessibility relation on the canonical state space. This is not case in the probabilistic framework, where strong axioms are needed to guarantee that. Fortunately, the non-standard structures permit huge simplifications, and one can devise an axiom system that essentially mimics the Minimal Epistemic Logic above described.

Minimal Probabilistic Logic
(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \to \psi$ infer ψ
(A1) $L_0\phi$
(A2) $L_a\top$
(A2+) $\neg L_a\bot$ $(a > 0)$
(A7) $L_a\phi \to L_b\phi$ $(b < a)$

The axiom's notation follows Heifetz & Mongin (2001) to facilitate comparison. Axioms (A2) and (A2+) reflect our semantic for $\top$ and $\bot$: the agent believes to maximal degree what he or she considers as necessarily true and does not believe to any degree what he or she considers as necessarily false. (A1) and (A7) reflect principles specific to the probabilistic case. Note that both bear on a single embedded formula ϕ: there is no doxastic reflection of a logical relation. They express something like a minimal metric of partial beliefs.

If $\models_{NSEPS} \phi$ means that ϕ is true in every non-standard explicit probabilistic structure and $\vdash_{MPL} \phi$ that ϕ is provable in the Minimal Probabilistic Logic, then we are ready to state our main result:

Theorem 1 (Completeness of MPL)

$$\models_{NSEPS} \phi \textit{ iff } \vdash_{MPL} \phi$$

Proof : see the Appendix.

1.4 Insights into the decision-theoretic case

We would like to end this paper by showing how to build choice models without logical omniscience, and which are the challenges raised by such a project.

1.4.1 Choice models without logical omniscience

The basic method to build a choice model without logical omniscience is to *substitute* one of our non-standard structures to the original doxastic model in the target choice model. We will now show how this could be done.

One might generically see models of choice under uncertainty as based on

- a state space S
- a consequence function $\mathfrak{C} : S \times A \rightarrow C$ where C is a set of consequences
- a utility function $u : C \rightarrow \mathbb{R}$
- a criterion of choice

To complete the choice model, one adds a distribution P on S for models of choice under probabilistic uncertainty, and a set $K \subseteq S$ of states compatible with the agent's beliefs under set-theoretic uncertainty.

To rigourously extend non-standard structures to choice models, one should translate the above described notions in a logical setting. But to give some insights, we will, on the contrary, import non-standard structures in the syntax-free framework of conventional decision theory. Let's have a look at the following, admittedly particular, target situation : an agent knows abstractly the consequence function $\mathfrak{C}$, but, because of limited computational capacities,he or she is not able, at the moment of choice, to perfectly infer from the choice function the consequence of each action at each possible state. One can think about a classic two-state example of insurance application [14]. The consequence function is

$\mathfrak{C}(s_1, x) = w - \pi x$

$\mathfrak{C}(s_2, x) = y + x$,

where x, the choice variable, is the amount of money spent in insurance, s_1 the state without disaster, w the wealth in s_1, s_2 the state with a disaster and y the subsequent wealth, and π the rate of exchange. In this case, a non logically omniscient with respect to the consequence function would be such that he or she ignores the value of $\mathfrak{C}$ for some arguments.

A simple way to model this target situation would be the following one. Let's consider *extended states* w, which are composed of a (primitive) state s and a local consequence function $\mathfrak{C}_w : A \rightarrow C$: $w = (s, \mathfrak{C}_w)$. The set of extended states is intended to represent the beliefs of the agent, including his or her logically imperfect beliefs. An extended state is standard if its local consequence function is conform to the (true) consequence function: $\mathfrak{C}_w(a) = \mathfrak{C}(a, s)$; if not, it is non-standard.

For instance, a logically imperfect agent could not know what is the consequence of action a in state s, thinking that it is possible that this consequence is c_i (let's say, the true one) or c_j. This situation would be modeled by building (at least) two extended states :

- $w_i = (s, \mathfrak{C}_{w_i})$ where $\mathfrak{C}_{w_i}(a) = c_i$, and
- $w_j = (s, \mathfrak{C}_{w_j})$ where $\mathfrak{C}_{w_j}(a) = c_j$

A perfect logician wouldn't have considered a possible state like w_j. On this basis, one can build choice models without the assumption of logical omniscience:

- in the case of choice under set-theoretic uncertainty, if one takes the maximin criterion, for a belief set $K \subseteq W$, the solution is :

$$Sol(A, S, W, C, \mathfrak{C}, u, K) = \arg\max_{a \in A} \min_{w \in K} u(\mathfrak{C}_w(a))$$

- in the case of choice under probabilistic uncertainty, if one takes the maximization of expected utility criterion, for a probability distribution P on W, the solution is :

$$Sol(A, S, W, C, \mathfrak{C}, u, P) = \arg\max_{a \in A} \sum_{w \in W} P(w).u(\mathfrak{C}_w(a))$$

1.4.2 Perspectives

From the decision theorists point of view, the substitution we have just described is only a first step. Two fundamental questions remains.

(a) First, there is the question of the axiomatization of the new choice models, that is closely linked with the behavioral implications of choice models without logical omniscience. In a recent paper, B. Lipman (Lipman (1999)) has remarkably tackled this issue, advocating a very similar approach. But the choice model he uses is quite specific (conditional expected utility), and one would like to compare choice models based on non-standard structures with the savagean benchmark.

More precisely, one would like to obtain a *representation theorem* à la Savage: define conditions on a preference relation $\succeq$ such that there exists (1) a space of extended states W, (2) a probability distribution P on W and (3) a utility function u such that the preference relation could be rationalized by the expected utility defined over preceding notions.

(b) Second, the non-standard choice models weakens only the cognitive assumptions of the (underlying) doxastic model. But there remains cognitive assumptions concerning the utility function and the choice criterion. In the approach we just described, we still assume that the agent is able to assign a precise utility to each consequence $c \in C$ and to calculate the solution to its choice criterion. Therefore, from the point of view of the bounded rationality programm, our proposition is strongly incomplete.

2 Tutorial 2 : Unawareness

2.1 Introduction

2.1.1 Intuitions and example

We studied in Tutorial 1 one the two main cognitive idealizations of current doxastic models. Let's turn now to the second of these idealizations, namely **full awareness**. In this introductory section, I would like to give some insights on the intuitive phenomena that are the target of the literature on (un)awareness. Unawareness is not less easy to characterize than lack of logical omniscience. Let's have a look at the explanations that can be given in one of leading contributions in the field, Modica & Rustichini (1999). According to them, there is unawareness exactly when

- there is "ignorance about the state space"
- "some of the facts that determine which state of nature occurs are not present in the subject's mind"
- "the agent does not know, does not know that she does not know, does not know that she does not know that she does not know, and so on..."

An example will probably make these insights clearer. Suppose that, for the next holiday, Pierre plans to rent a house. And let's assume that, from the modeler point of view, there are three main factors that are relevant for his choice:

- p: the house is no more than 1 km far from the sea
- q: the house is no more than 1 km far from a bar
- r: the house is no more than 1 km far from an airport

In this scenario, it seems that there is an intuitive distinction between the two following doxastic situations. Situation (i): Pierre simply ignores r : he doesn't know whether there is an airport no more than 1 km far from the house - there are both r-states and $\neg r$-states which are epistemically accessible from his point of view. Situation (ii): Pierre doesn't ask to himself : 'is there an airport no more than 1 km far from the house?"

In the situation (i), Pierre considers the possibility that r but is not in a position to believe that it is true. In situation (ii), the possibility that r is out of his mind.[15] In our example, the state space as seen by the modeler could be this one:[16]

pqr	$p\neg qr$	$pq\neg r$	$p\neg q\neg r$
$\neg pqr$	$\neg p\neg qr$	$\neg pq\neg r$	$\neg p\neg q\neg r$

On the other hand, our discussion suggests that the "subjective state space" as seen by Pierre could be this one:

pq	$p\neg q$
$\neg pq$	$\neg p\neg q$

Note that Pierre's subjective state space is obtained by fusion of some states in the modeler's state space: a fusion between states that differ only in the truth value they assign to the formula the agent is unaware of - namely r. To put it in other words, the subjective state space is a less fine-grained state space.[17]

2.1.2 Principles

Let's shift from examples to principles. What are the intuitive properties of (un)awareness ? Here are some intuitive properties that may be found in the literature (in what follows$A\phi$ means "the agent is aware that ϕ and $U\phi$ "the agent is unaware that ϕ):

$A\phi \leftrightarrow A\neg\phi$	(Symmetry)
$A(\phi \wedge \psi) \leftrightarrow A\phi \wedge A\psi$	
$A\phi \leftrightarrow AA\phi$	(Self-Reflection)
$U\phi \rightarrow UU\phi$	(U-introspection)
$U\phi \rightarrow \neg B\phi \wedge \neg B\neg B\phi$	(Plausibility)
$U\phi \rightarrow (\neg B)\phi^n \ \forall n \in N$	(Strong Plausibility)
$\neg BU\phi$	(BU-introspection)

Those properties are attractive. Unfortunately, it has been recognized since the end of the nineties that, in a framework like epistemic logic, it is not possible to devise a non-trivial (un)awareness operator that satisfies most of these properties. In particular, in a ground-breaking paper, Dekel, Lipman & Rustichini (1998) show that it is impossible to have

(i) a non-trivial awareness operator which satisfies Plausibility, U-introspection and BU-introspection

(ii) a belief operator which satisfies either Necessitation or Monotonicity

2.2 (Un)awareness in epistemic logic

How to circumvent the issue raised by the triviality results as the one mentioned above ? The literature has investigated two main ways of characterizing the (un)awareness operator :

(i) endogenous characterization: in this case, (un)awareness is defined in terms of beliefs. The basic idea is that to be aware of ϕ is to believe that ϕ or to believe that one does not believe that ϕ :

$$\mathcal{M}, s \models A\phi \Leftrightarrow \mathcal{M}, s \models B\phi \vee B\neg B\phi$$

This approach is the one endorsed by Modica & Rustichini (1999) and Heifetz, Meier & Schipper (2006).

(ii) exogenous characterization: in this case, (un)awareness is a primitive operator. This approach is followed by Fagin and Halpern with their awareness© structures (see above and Fagin & Halpern (1988), Fagin et al. (1995), Halpern (2001)):

$$\mathcal{M}, s \vDash A\phi \Leftrightarrow \phi \in \mathcal{A}(s)$$

where $\mathcal{A} : S \to \wp(\mathcal{L}(At))$ Fagin & Halpern (1988), Halpern (2001)

The awareness© structures have been presented in Tutorial 1. I will focus on the model of Modica & Rustichini (1999), the GSM structures.

Definition 16
A GSM structure is a t-uple $\mathcal{M} = (S, S', \pi, R, \rho)$

- (i) S *is a state space*
- (ii) $S' = \bigcup_{X \subseteq At} S'_X$ *(where* S'_X *are disjoint) is a (non-standard) state space*
- (iii) $\pi : At \times S \to \{0, 1\}$ *is a valuation for* S
- (iv) $R : S \to \wp(S')$ *is an accessibility correspondence*
- (v) $\rho : S \to S'$ *is a projection s.t. (1) if* $\rho(s) = \rho(t) \in S'_X$*, then (a) for each atomic formula* $p \in X$*,* $\pi(s, p) = \pi(t, p)$ *and (b)* $R(s) = R(t)$ *and (2) if* $\rho(s) \in S'_X$*, then* $R(s) \subseteq S'_X$

Note that, in a GSM structure, each state s is associated to a subjective state space S'_X. Furthermore, one can extend R and π to the whole state space with π^*: if $s' \in S'_X$, then $\pi^*(s', p) = 1$ iff (a) $p \in X$ and (b) for all $s \in \rho^{-1}(s'), \pi(s, p) = 1$.

Definition 17
The ***satisfaction relation*** *for GSM-structures is defined for each* $s^* \in S^* = S \cup S'$ *(Halpern's 2001 version):*

- (i) $\mathcal{M}, s^* \vDash p$ *iff* $\pi^*(s^*, p) = 1$
- (ii) $\mathcal{M}, s^* \vDash \phi \wedge \psi$ *iff* $\mathcal{M}, s^* \vDash \phi$ *and* $\mathcal{M}, s^* \vDash \psi$
- (iii) $\mathcal{M}, s^* \vDash \neg\phi$ *iff* $\mathcal{M}, s^* \nvDash \phi$ *and either* $s^* \in S$*, or* $s^* \in S'_X$ *and* $\phi \in \mathcal{L}(X)$
- (iv) $\mathcal{M}, s^* \vDash B\phi$ *iff for each* $t^* \in R^*(s^*)$*,* $\mathcal{M}, t^* \vDash \phi$

From the point of view of the general aim of this Tutorial, there is one thing that is necessary to remark at this point: *GSM-structures are nothing but a special kind of non-standard structures* as defined in Tutorial 1. I will end the presentation of GSM structures with an example to illustrate how works this model.

Example 7
Let's suppose that the (objective) state space is based on the finite language $At = \{p, q, r\}$*. From the modeler point of view, the possible states are these one:*

$\boldsymbol{pqr}$	$p\neg qr$	$pq\neg r$	$p\neg q\neg r$
$\neg pqr$	$\neg p\neg qr$	$\neg pq\neg r$	$\neg p\neg q\neg r$

Let's consider a pointed GSM structure where the actual state is the state $s = pqr$ *(in bold) and let's assume that Pierre believes that* p*, does not know whether* q *and is unaware of* r*. Pierre's non-standard state space* $S_{p,q}$ *and accessibility correspondence in* s *may be represented as follows (accessible states in bold):*

$\boldsymbol{pq}$	$\boldsymbol{p\neg q}$
$\neg pq$	$\neg p\neg q$

This situation can be explicitely articulated with the concepts defining GSM-structures:
- the state pqr *in projected in the state* pq*:* $\rho(pqr) = pq$
- $R(pqr) \subseteq S_{pq}$
The defining properties of the GSM-structures imply the following propositions : if pqr *and* $pq\neg r$ *are projected in* pq*,* pqr *and* $pq\neg r$ *agree on* p *and* q*; and if* pqr *and* $pq\neg r$ *are projected in* pq*,* $R(pqr) = R(pq\neg r)$*.*

2.3 Unawareness and partiality

One of the crucial features of the definition of satisfaction for GSM structures is related to the case of negation (iii) : the truth-conditions for negated formulas introduces *partiality*. As a matter of fact, if $p \notin X$ and $s^* \in S'_X$ then neither $\mathcal{M}, s^* \models p$ nor $\mathcal{M}, s^* \models \neg p$. I will use $\mathcal{M}, s^* \Uparrow \phi$ to denote the fact that the formula ϕ is undefined at s^*, and $\mathcal{M}, s^* \Downarrow \phi$ to denote the fact that it is defined. One can show that the following fact is true :

$$\mathcal{M}, s^* \Downarrow \phi \text{ for } s^* \in S'_X \text{ iff } \phi \in \mathcal{L}(X)$$

As I said before, in GSM structures, the (un)awareness operator is endogeneous : it is defined in terms of beliefs. Heifetz et al. (2006) proceeds in the same way in their multi-agent framework. It is significant that both models are in general restricted to the partitional case, i.e. to the case where the accessibility relations are partitional. Actually, their characterization of (un)awareness seems to me satisfactory *but only under this restriction.* To put it in other words, when the relation of accessibility is no longer assumed to be partitional, this endogenous characterization is no longer satisfactory: it is not robust under change of the accessibility relation's properties.

More precisely, in the general case, it seems to me that

- it is plausible that if Pierre is unaware of ϕ, he doesn't believe that ϕ nor believe that he doesn't believe that ϕ $(U\phi \rightarrow (\neg B\phi \wedge \neg B \neg B\phi))$; but

- it is *not* plausible that if Pierre doesn't believe that ϕ nor believe that he doesn't believe that ϕ, he is necessarily unaware of ϕ $((\neg B\phi \wedge \neg B \neg B\phi) \rightarrow U\phi)$

The following example will support my claims.

Example 8
Suppose that

- *the actual state s is projected in $s_1 \in S'_{\{p,q\}}$*
- $R(s_1) = \{s_2, s_3\}$ *(hence $s_2, s_3 \in S'_{\{p,q\}}$ as well)* ; $R(s_2) = \{s_2\}$; $R(s_3) = \{s_3\}$
- $\mathcal{M}, s_2 \models \neg p$, *hence* $\mathcal{M}, s_2 \models \neg Bp \wedge B\neg p$
- $\mathcal{M}, s_3 \models p$, *hence* $\mathcal{M}, s_3 \models Bp$

In this structure (which is not partitional), it is the case that $\mathcal{M}, s \models \neg Bp \wedge \neg B \neg Bp$ hence $\mathcal{M}, s \models Up$. But, for instance, $\mathcal{M}, s \models B(B\neg p \vee Bp)$. It is my contention that the joint truth of $\mathcal{M}, s \models Up$ and $\mathcal{M}, s \models B(B\neg p \vee Bp)$ is highly implausible.

In reaction to this issue, my proposal is the following one: to keep the underlying GSM structure, but to change the definition of (un)awareness. On which basis to define the (un)awareness operator ? It seems to me that the *partiality* stressed above is an intuitive and reliable guide. If we consider the initial example the intuition would be that the possible states that Pierre conceives do not "answer" to the question "Is is true that r ?" whereas they answer to the questions "Is it true that p?" and "Is it true that q ?". In other words, the possible states that Pierre conceives make true neither r nor $\neg r$. This intuition leads to a semantic characterization of (un)awareness in termes of *partiality* :

$$\mathcal{M}, s \models A\phi \text{ iff } \exists t \in R(s),\ \mathcal{M}, t \Downarrow \phi$$

Let's call a **P-GSM structure** a GSM structure where the truth conditions of the unawareness operator are given in terms of partiality. P-GSM structures are my proposal for dealing with (un)awareness.

Halpern (2001) relates GSM structures and awareness$^{©}$ structures. Actually, one obtains a still closer connection with P-GSM structures. Let's say that an awareness$^{©}$ structure $\mathcal{M} = (S, R, \mathcal{A}, \pi)$ is *propositionally determined* (pd) if (1) for each state s, $\mathcal{A}(s)$ is generated by some atomic formulas $X \subseteq At$ i.e. $\mathcal{A}(s) = \mathcal{L}(X)$ and (2) if $t \in R(s)$, then $\mathcal{A}(s) = \mathcal{A}(t)$.

Proposition 6 (Halpern 2001 Thm 4.1)

1. *For every pd awareness$^{©}$ structure $\mathcal{M}$ there exists a P-GSM structure $\mathcal{M}'$ based on the same state space S and the same valuation π s.t. for all formulas $\phi \in \mathcal{L}^{BA}(At)$ and each possible state s*

   $\mathcal{M}, s \models_{a^{©}} \phi$ *iff* $\mathcal{M}', s \models_{P-GSM} \phi$

2. *For every P-GSM structure $\mathcal{M}$ there exists a awareness$^{©}$ structure $\mathcal{M}'$ based on the same state space S and the same valuation π s.t. for all formulas $\phi \in \mathcal{L}^{BA}(At)$ and each possible state s*
   $\mathcal{M}, s \models_{P-GSM} \phi$ *iff* $\mathcal{M}', s \models_{a^{©}} \phi$

One corollary of this Proposition is that the axiom system K_X devised by Halpern (2001) is sound and complete for P-GSM structures.

system K_X (Halpern 2001)
(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ
(A0) $B\phi \Rightarrow A\phi$
(K) $B\phi \wedge B(\phi \rightarrow \psi) \rightarrow B\psi$
(Gen) From ϕ infer $A\phi \rightarrow B\phi$
(A1) $A\phi \leftrightarrow A\neg\phi$
(A2) $A(\phi \wedge \psi) \leftrightarrow (A\phi \wedge A\psi)$
(A3) $A\phi \leftrightarrow AA\phi$
(A4) $AB\phi \leftrightarrow A\phi$
(A5) $A\phi \rightarrow BA\phi$
(Irr) If no atomic formulas in ϕ appear in ψ, from $U\phi \rightarrow \psi$ infer ψ

2.4 The probabilistic case

In Tutorial 1 on logical omniscience, we have seen that one the main advantages of the impossible states approach is that it can be straightforwardly extended to the probabilistic case. In this section, we will investigate a similar extension for the modeling of (un)awareness.

2.4.1 Probabilistic (un)awareness, first attempt

Let's begin with the most simple probabilistic extension of P-GSM structure :

Definition 18
Definition *: an* ***P-GSM explicit probabilistic structure*** *for $\mathcal{L}^{LA}(At)$ is a t-tuple $\mathcal{M} = (S, S', \pi, P, \rho)$ where*

(i) *S is a state space*

(ii) $S' = \bigcup_{\Phi \subseteq At} S'_\Phi$ *(where S'_Φ are disjoint) is a state space*

(iii) $\pi : At \times S \rightarrow \{0,1\}$ *is a valuation for S*

(iv) $P : S \rightarrow \Delta(S')$

(v) $\rho : S \rightarrow S'$ *is a projection s.t. (1) if $\rho(s) = \rho(t) \in S'_\Phi$, then (a) for each atomic formula $p \in \Phi$, $\pi(s,p) = \pi(t,p)$ and (b) $P(s) = P(t)$ and (2) if $\rho(s) \in S'_\Phi$, then $Supp(P(s)) \subseteq S'_\Phi$*

The satisfaction condition for the probabilistic operator $L_a\phi$ is the usual one :

$$\mathcal{M}, s \models L_a\phi \Leftrightarrow P(s)([[\phi]]) \geq a$$

How does such a probabilistic system behave ? *Prima facie*, its behavior is quite satisfactory since, for instance, the following proposition may be established:

Proposition 7
For all P-GSM explicit probabilistic structure $\mathcal{M}$ and all standard state s, unawareness precludes positive probability:

$\mathcal{M}, s \models U\phi \rightarrow \neg L_a\phi$ *for* $a > 0$

$\mathcal{M}, s \models U\phi \rightarrow \neg L_a\neg\phi$ *for* $a > 0$

$\mathcal{M}, s \models \neg L_a U\phi$ *for* $a > 0$

But some specifically probabilistic features induce highly counter-intuitive consequences :

Proposition 8
For all P-GSM explicit probabilistic structure $\mathcal{M}$ and all standard state s, the following formulas are satisfied:

$\mathcal{M}, s \vDash U\phi \rightarrow L_0\phi$

$\mathcal{M}, s \vDash U\phi \rightarrow L_0\neg\phi$

$\mathcal{M}, s \vDash U\phi \rightarrow L_1 L_0\phi$

2.4.2 Probabilistic unawareness, second attempt

The last Proposition shows that our first attempt to extend P-GSM structures to the probabilistic case is a failure: in this setting, if an agent is *un*aware of a formula ϕ, he believes to degree 1 that he believes at least to degree 0 that ϕ. It seems that we have lost the basic intuition according to which an agent is unaware of a formula if the possibility expressed by this formula is "out of" his or her epistemic space.

Hopefully, we can improve this first attempt by modifying slightly not the definition of P-GSM explicit probabilistic structures but the satisfaction condition of the operator $L_a\phi$. This modification consists in a requirement: for an agent to believe at least to some degree that ϕ, it is necessary that ϕ is defined in the possible states that are accessible to him or her. Formally:

$$\mathcal{M}, s \vDash L_a\phi \Leftrightarrow P(s)([[\phi]]) \geq a \textbf{ and } \mathcal{M}, \rho(s) \Downarrow \phi$$

With this modification, the probabilistic logic delivers much more intuitive properties. Actually, one can show that all the properties that were intuitive in the epistemic case can be given a valid probabilistic counterpart:

Proposition 9
For all P-GSM explicit probabilistic structure $\mathcal{M}$ and all standard state s, the following formulas are satisfied:

$A\phi \leftrightarrow A\neg\phi$	*(Symmetry)*
$A\phi \leftrightarrow AA\phi$	*(Self-Reflection)*
$U\phi \rightarrow UU\phi$	*(U-introspection)*
$U\phi \rightarrow \neg L_a\phi \wedge \neg L_a\neg L_a\phi$	*(Plausibility)*
$U\phi \rightarrow (\neg L_a)^n\phi\ \forall n \in N$	*(Strong Plausibility)*
$\neg L_a U\phi$	*($L_a U$-introspection)*
$L_0\phi \leftrightarrow A\phi$	

2.5 Perspectives

There is two our mind four main issues to be explored concerning (un)awareness and more specifically probabilistic (un)awareness :

(1) axiomatizing probabilistic (un)awareness
(2) becoming aware
(3) multi-agent unawareness
(4) applications to decision theory and game theory

I won't say anything concerning issues (3) and (4). Issue (1) consists basically to achieve with respect to P-GSM explicit probabilistic structures what we have achieved with respect to non-standard explicit probabilistic structures: to devise an axiom system and prove a completeness theorem. From a conceptual point of view the trickiest issue is undoubtedly issue (2): how to model the fact that an agent becomes aware of something ? One could informally think of the following procedure: if an agent becomes aware of r, then his or her subjective state space is enriched. Typically, he shifts from this first subjective state space:

pq	$p\neg q$
$\neg pq$	$\neg p\neg q$

to this second, more fine-grained, subjective state space:

pqr	$p\neg qr$	$pq\neg r$	$p\neg q\neg r$
$\neg pqr$	$\neg p\neg qr$	$\neg pq\neg r$	$\neg p\neg q\neg r$

The hard question is this one: how do evolve the beliefs of the agent from one state space to the other? Things may look simple in the epistemic case (full beliefs). If a state was epistemically accessible initially, let's say pq, then the two possible states obtained by splitting it (namely pqr and $pq\neg r$) are epistemically possible after the agent has became aware of r. This principle of belief change may be less innocuous that it seems, but in any case the crucial difficulty comes with the probabilistic case. For suppose that we follow the same method. And let $P(pq)$ be the initial probability on the state pq. Can we infer from $P(pq)$ the new probabilities on states pqr and $pq\neg r$? It seems to me that the answer has to be negative: there is hardly one principled way of re-allocating probabilities on such states. The naive rule according to which the weight of state pq should be equally divided and re-allocated to the two worlds won't do the job.

3 Appendix

Proof of Proposition 5

The proof deals only with the case of submodularity ; the other is symetric. If $[[\phi]]^*$ and $[[\psi]]^*$ are disjoint, then by hypothesis $[[\phi\wedge\psi]]^* = \emptyset$. Therefore $CP(\phi\vee\psi) = CP(\phi)+CP(\psi)-CP(\phi\wedge\psi)$.

It follows from the definition that if $\mathcal{M}, s \vDash \psi\wedge\phi$, then $\mathcal{M}, s \vDash \psi$ and $\mathcal{M}, s \vDash \phi$ (the converse does not hold). In other words,

(1) $[[\phi\wedge\psi]]^* \subseteq [[\phi]]^* \cap [[\psi]]^*$.

This implies that

(2) $P([[\phi\wedge\psi]]^*) \leq P([[\phi]]^* \cap [[\psi]]^*)$.

Since $\mathcal{M}$ is $\vee$-standard, $P([[\phi\vee\psi]]^*) = P([[\phi]]^*) + P([[\phi]]^*) - P([[\phi]]^* \cap [[\psi]]^*)$. By (2), it follows from this that

$P([[\phi\vee\psi]]^*) \leq P([[\phi]]^*) + P([[\phi]]^*) - P([[\phi\wedge\psi]]^*)$. ■

Proof of Theorem 1

($\Rightarrow$). Soundness is easily checked and is left to the reader.

($\Leftarrow$). We have to show that $\models_{NSEPS} \phi$ implies $\vdash_{LPM} \phi$. First, let's notice that the Minimal Probabilistic Logic (MLP) is a "modal logic" (Blackburn, de Rijke & Venema (2001), 191): a set of formulas (1) that contains every propositional tautologies, (2) that is closed by *modus ponens* and uniform substitution. One can then apply the famous Lindenbaum Lemma.

Definition 19

(i) A formula ϕ is ***deducible*** *from a set of formulas Γ, symbolized $\Gamma \vdash \phi$, if there exists some formulas $\psi_1, ..., \psi_n$ in Γ s.t. $\vdash (\psi_1 \wedge ... \wedge \psi_n) \rightarrow \phi$.*

(ii) A set of formulas Γ is ***Λ-consistent*** *if it is false that $\Gamma \vdash_\Lambda \bot$*

(iii) A set of formulas Γ is ***maximally Λ-consistent*** *if (1) it is Λ-consistent and (2) if it is not included in a Λ-consistent set of formulas.*

Lemma 1 (Lindenbaum Lemma)
If Γ is a set of Λ-consistent formulas, then there exists an extension Γ^+ of Γ that is maximally Λ-consistent.

Proof. See for instance Blackburn et al. (2001), p.199.

Definition 20
Let $\phi \in L$; the language associated with ϕ, $\mathfrak{L}_\phi$ is the smallest sub-language that

(i) contains $\phi, \bot$ et $\top$;

(ii) is closed under sub-formulas

(iii) is closed under the symbol $\sim$ defined as follows : $\sim \chi := \psi$ if $\chi := \neg\psi$ and $\sim \chi := \neg\chi$ if not.[18]

In the language $\mathfrak{L}_\phi$, one can define the analogon of the maximally Λ-consistent sets.

Definition 21
An atom *is a set of formulas in formules de $\mathfrak{L}_\phi$ which is maximally Λ-consistent. $At(\phi)$ is the set of atoms.*

Lemma 2
For every atom Γ,

(i) there exists an unique extension of Γ in L, symbolized Γ^+, that is maximally Λ-consistent ;

(ii) $\Gamma = \Gamma^+ \cap \mathfrak{L}_\phi$

Proof. (i) is an application of Lindenbaum Lemma. (ii) is implied by the fact that Γ is maximally coherent. Suppose that there exists a formula ψ from $\mathfrak{L}_\phi$ in Γ^+ but not in Γ, then Γ^+ woud be inconsistent, what is excluded by hypothesis.■

Starting from atoms, one may define the analogon of canonical structures, *i.e.* structures where (standard) states are sets of maximally Λ-consistent formulas. In the same way, we will take as canonical standard state space the language's $\mathfrak{L}_\phi$ atoms.

The hard stuff is the definition of the probability distributions. The aim is to make true in s_Γ every formula $L_a\chi$ in the atom Γ associated with the state s_Γ. To do that, it is necessary that $P(s_\Gamma, \chi) \geq a$; this is guaranted if one takes for $P(s_\Gamma, \chi)$ the number b^* s.t. $b^* = \max\{b : L_b\chi \in \Gamma\}$. This can easily be done with non-standard states. It will be the case if (1) the support of $P(s_\Gamma, .)$ is included in the set of non-standard states, (2) $P(s_\Gamma, .)$ is equiprobable and (3) there is a proportion b^* of states that make χ true.

Suppose that $I(\Gamma)$ is the sequence of formulas in Γ that are prefixed by a doxastic operator L_a ; for every formula, one can rewrite $b^*(\chi)$ as p_i/q_i. Define $q(\Gamma) = \prod_{i \in I} q_i$; $q(\Gamma)$ will be the set of non-standard states in which $P(s_\Gamma, .)$ will be included. If the i-st formula is χ, suffice it to stipulate that χ in the first $p_i \times \prod q_{-i}$ states. One may check that the proportion of states χ is true is p_i/q_i.

Definition 22
The ***ϕ-canonical structure*** *is the structure $\mathcal{M}_\phi(S_\phi, S'_\phi, \pi_\phi, \vDash_\phi, P_\phi)$ where*

(i) $S_\phi = \{s_\Gamma : \Gamma \in At(\phi)\}$

(ii) $S'_\phi = \bigcup_{\Gamma \in At(\phi)} q(\Gamma)$

(iii) for all standard state, $\pi_\phi(p, s_\Gamma) = 1$ iff $p \in s_\Gamma$

(iv) for all non-standard state s, $\mathcal{M}_\phi, s \vDash_\phi \psi$ iff, if $s \in q(\Gamma)$ and ψ is the i-st formula prefixed by a doxastic operator in Γ, then s is in the $p_i \times \prod q_{-i}$ first states of $q(\Gamma)$

(v) $P_\phi(s_\Gamma, .)$ is an equiprobable distribution on $q(\Gamma)$

As expected, the ϕ-canonical structure satisfies the Truth Lemma.

Lemma 3 (Truth Lemma)
For every atom Γ, $\mathcal{M}_\phi, s_\Gamma \models \psi$ *iff* $\psi \in \Gamma$

Proof. The proof proceeds by induction on the length of the formula.

(a) $\psi := p$; follows directly from the definition of π_ϕ.

(b) $\psi = \psi_1 \vee \psi_2$; by definition, $\mathcal{M}_\phi, s \models_\phi \psi$ iff $\mathcal{M}_\phi, s \models_\phi \psi_1$ or $\mathcal{M}_\phi, s \models_\phi \psi_2$. Case (b) will be checked if one shows that $\psi_1 \vee \psi_2 \in \Gamma$ iff $\psi_1 \in \Gamma$ or $\psi_2 \in \Gamma$. Let's consider the extension Γ^+ de Γ ; one knows that $\psi_1 \vee \psi_2 \in \Gamma^+$ iff $\psi_1 \in \Gamma^+$ or $\psi_2 \in \Gamma^+$. But $\Gamma = \Gamma^+ \cap \mathfrak{L}_\phi$ and ψ, ψ_1 and $\psi_2 \in \mathfrak{L}_\phi$. It follows that $s_\Gamma \models \psi_1 \vee \psi_2$ iff $\psi_1 \vee \psi_2 \in \Gamma$.

(c) $\psi = \neg\chi$. $\mathcal{M}_\phi, s \models_\phi \neg\chi$ iff $\mathcal{M}_\phi, s \not\models_\phi \chi$ iff (by induction hypothesis) $\chi \notin \Gamma$. Suffice it to show that $\chi \notin \Gamma$ iff $\neg\chi \in \Gamma$. (i) Let's suppose that $\chi \notin \Gamma$; χ is in $\mathfrak{L}_\phi$ hence, given the properties of maximally Λ-consistent sets, $\neg\chi \in \Gamma^+$. And since $\Gamma = \Gamma^+ \cap \mathfrak{L}_\phi$, $\neg\chi \in \Gamma$. (ii) Let's suppose that $\neg\chi \in \Gamma$; Γ is coherent, therefore $\chi \notin \Gamma$.

(d) $\psi = L_a\chi$; by definition $s_\Gamma \models L_a\chi$ iff $P(s_\Gamma, \chi) \geq a$. (i) Let's suppose that $P(s_\Gamma, \chi) \geq a$; then $a \leq b^*$ where $b^* = max\{b : L_b\chi \in \Gamma$ since by definition of the canonical distribution, $P(s_\Gamma, \chi) = b^*$. Now, let's consider the extension Γ^+ : clearly, $L_b^*\chi \in \Gamma^+$. In virtue of axiom (A7) and of the closure under *modus ponens* of maximally Λ-consistent sets, $L_a \in \Gamma^+$. Given that by hypothesis $L_a\chi \in \mathfrak{L}_\phi$, this implies that $L_a\chi \in \Gamma$. (ii) Let's suppose that $L_a\chi \in \Gamma$; then $a \leq b^*$ hence $P(s, \chi) \geq a$.■

To prove completeness, we need a last lemma.

Lemma 4
Let $At(\phi)$ *the set of atoms in* $\mathfrak{L}\phi$ *;*
$At(\phi) = \{\Delta \cap \mathfrak{L}\phi : \Delta$ *is maximally coherent* $\}$.

Proof. $At(\phi) \subseteq \{\Delta \cap \mathfrak{L}\phi : \Delta$ is maximally coherent $\}$ follows from a preceding lemma. Let Γ^+ a maximally consistent set and $\Gamma = \Gamma^+ \cap \mathfrak{L}_\phi$. We need to show that Γ is maximally consistent in $\mathfrak{L}_\phi$. First Γ is consistent ; otherwise, Γ^+ would not be. Then, we need to show that Γ is maximal, *i.e.* that for every formula $\psi \in \mathfrak{L}_\phi$, if $\Gamma \cup \{\psi\}$ is consistent, then $\psi \in \Gamma$. Let ψ such a formula. Let's recall that Γ^+ is maximally consistent. Either $\psi \in \Gamma^+$ and then $\psi \in \Gamma$; or $\neg\psi \in \Gamma^+$ (elementary property of maximally consistent sets) and, if $\psi := \neg\chi$, $\chi \in \Gamma^+$ as well. Hence, by definition of $\mathfrak{L}_\phi$, χ or $\neg\neg\chi \in \Gamma$. But this is not compatible with the initial hypothesis according to which $\Gamma \cup \{\phi\}$ is consistent.■

We can now finish the proof : let ϕ a LPM-consistent formula. Then, there exists a maximally LPM-consistent set Γ^+ which contains ϕ. Let $\Gamma = \Gamma^+ \cap \mathfrak{L}_\phi$. ϕ is in Γ therefore by the Truth Lemma, ϕ is true in state s_Γ of the ϕ-canonical structure. Then ϕ is satisfiable. ■

Notes

[1]One important exception is Lipman (1999).

[2]This formulation (the so-called Backus-Naur Form) means, for instance, that the propositional variables are formulas, that if ψ is a formula, $\neg\psi$ too, and so on.

[3]We have contributed to this industry by defending the use of substructural logics in **?** ; this setting is not tractable enough for the aim of the current paper.

[4]This reccuring example is not chosen for its cognitive realism, but because it makes the comparison of different solutions easy.

[5]What we call "awareness© structures" is called in the original paper "logic of general awareness".

[6]For a more extensive defense of the solution, see Hintikka (1975) or, more recently, Barwise (1997).

[7]In the paper, to avoid complications that are unnecessary for our purpose, we suppose that S is finite and that P is defined on $\wp(S)$.

[8]"Elementary" because there is no doxastic operator in the object-language.

[9]See Fagin & Halpern (1991). For a recent reference on logical formalization of probabilistic reasoning, see Halpern (2003).

[10]Note that $CP(\phi) = r$ is in the meta-language, not in the object-language.

[11]A similar idea has been defended a long time ago by I. Hacking who talks about "personal possibility", by contrast with "logical possibility". We won't develop the point here, but this contribution can be seen as a formalization of Hacking's insights.

[12]Economists are leading contributors to the study of explicit probabilistic structures because they correspond to the so-called *type spaces* used in games of incomplete information, in the same way that Kripke structures (with R as an equivalence relation). See Aumann & Heifetz (2002).

[13]Note that another language is used by J. Halpern in Fagin & Halpern (1991) or Halpern (2003).

[14]From Lippman & McCall (1981).

[15]The idea is quite close to the one expressed by Savage (1954) with the states of so-called "small worlds".

[16]Here and beyond, I name the possible states by the atomic formula or negation of atomic formula that are true in the states.

[17]Once again, the idea is already present in Savage (1954): "...a smaller world is derived from a larger by neglecting some distinctions between states"

[18]See Blackburn et al. (2001), p.242.

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