Impossible States at Work

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introduction

- decision theory has three building blocks :
  - a model of beliefs (doxastic model)
  - a model of desires (axiological model)
  - a criterion of choice, which, given beliefs and desires, selects the "right" actions

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  - a model of beliefs (doxastic model)
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  - a criterion of choice, which, given beliefs and desires, selects the "right" actions
- full beliefs:
  - the classical model is epistemic logic
  - it is well known that epistemic logic suffers from **logical omniscience (LO)** : closure of beliefs under logical consequence, substituability of logically equivalent formulas, etc.
  - putative solutions to LO : neighborhood structures, awareness structures, non-standard (impossible) states structures, etc.

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  - mainstream decision theory is based on a model of partial beliefs
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  - it is rarely recognized that probabilistic models of beliefs suffer from LO as well
- decision theory inherits the cognitive idealizations of its doxastic models
- **Question**: how to weaken LO in models of partial beliefs ?

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- main claim: non-standard structures are the best candidate to this extension
    1. what are probabilistic counterparts of non-standard structures?
    2. what are the properties of these probabilistic counterparts?
    3. is there a justification for the use of these probabilistic counterparts?

Outline of the talk

Kripke structures

- **Definition** : The set of **formulas** of an epistemic propositional language $\mathcal{L}B(At)$ based on a set At of propositional variables $Form(\mathcal{L}B(At))$, is defined by

$$\phi ::= p|\neg\phi|\phi \vee \psi|B\phi$$

where $p \in At$.

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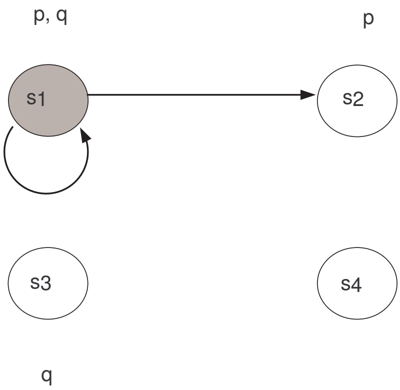
  where $p \in At$.
- **Definition** : let $\mathcal{L}B(At)$ an epistemic propositional language ; a **Kripke structure** for $\mathcal{L}B(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, R)$ where
  - (i) S is a state space,
  - (ii) $\pi : At \times S \rightarrow \{0, 1\}$ is a valuation
  - (iii) $R \subseteq S \times S$ is an accessibility relation

- **Definition** : $\bar{\pi}$, called the **satisfaction relation**, extends π to every formula of the langage according to the following conditions :
  - $\bar{\pi}(s, p) = \pi(s, p)$ if $p \in At$
  - $\bar{\pi}(s, \phi \vee \psi) = 1$ iff $\bar{\pi}(s, \phi) = 1$ or $\bar{\pi}(s, \psi) = 1$
  - $\bar{\pi}(s, \neg\phi) = 1$ iff $\bar{\pi}(s, \phi) = 0$
  - $\bar{\pi}(s, B\phi) = 1$ iff $\forall s'$ s.t. sRs', $\bar{\pi}(s, \phi) = 1$ (= ***possible-state analysis of belief*** = to believe something is to exclude that it could be false)

epistemic logic, example



Pierre believes that p, hence that $p \vee q$.

logical omniscience

- **Definitions** :
  (i) the **proposition** expressed by ϕ, or **informational content** of ϕ is $[[\phi]]_{\mathcal{M}} = \{s : \bar{\pi}(\phi, s) = 1\}$
  (ii) ϕ $\mathcal{M}$**-implies** ψ if $[[\phi]]_{\mathcal{M}} \subseteq [[\psi]]_{\mathcal{M}}$
  (iii) ϕ and ψ are $\mathcal{M}$**-equivalent** if $[[\phi]]_{\mathcal{M}} = [[\psi]]_{\mathcal{M}}$

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- **Proposition**: for all Kripke structure $\mathcal{M}$,
  - **Deductive monotony** : if ϕ $\mathcal{M}$-implies ψ, then $B\phi$ $\mathcal{M}$-implies $B\psi$
  - **Intensionality** : if ϕ and ψ are $\mathcal{M}$-equivalent, then $B\phi$ and $B\psi$ are $\mathcal{M}$-equivalent

neighborhood structures

- **Definition** : a **neighborhood structure** is a 3-tuple $\mathcal{M} = (S, \pi, V)$ where
  (i) S is a state space,
  (ii) $\pi : At \times S \to \{0, 1\}$ is a valuation,
  (iii) $V : S \to \wp(\wp(S))$, called the agent's **neighborhood system**, associates to every state a set of propositions.

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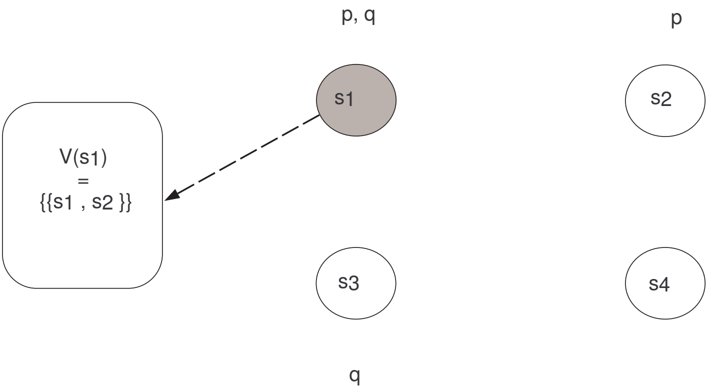
- New doxastic satisfaction condition :

$$\bar{\pi}(B\phi, s) = 1 \text{ iff } [[\phi]]_{\mathcal{M}} \in V(s)$$

neighborhood structures, example



Pierre believes that p but not that $p \vee q$ since $p \vee q \notin V(s_1)$.

neighborhood structures, axiomatization

System E (Chellas, 1980)

(PROP) Instances of propositional tautologies

(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

(RE) From $\phi \leftrightarrow \psi$ infer $B\phi \leftrightarrow B\psi$

awareness structures

- **Definition** : an **awareness structure** is a 4-tuple (S, π, R, A) where
  (i) S is a state space,
  (ii) $\pi : At \times S \rightarrow \{0, 1\}$ is a valuation,
  (iii) $R \subseteq S \times S$ is an accessibility relation,
  (iv) $A : S \rightarrow Form(\mathcal{L}B(At))$ is a function which maps every state in a set of formulas ("awareness set").

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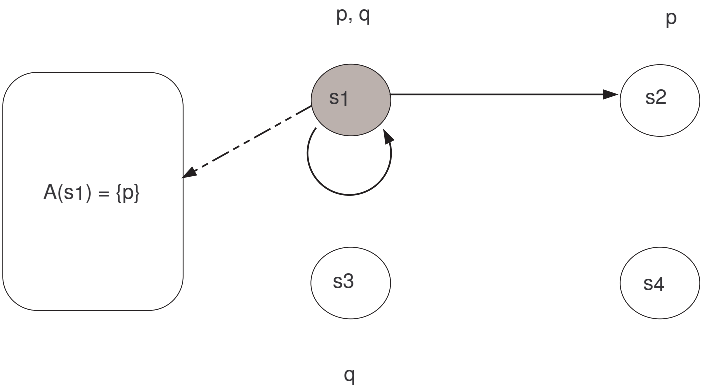
- New doxastic satisfaction condition :

$$\bar{\pi}(B\phi, s) = 1 \text{ iff } \forall s' \text{ s.t. } sRs', s' \in [[\phi]]_{\mathcal{M}} \text{ and } \phi \in A(s)$$

awareness structures, example



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awareness structures, axiomatization

Minimal Epistemic Logic (FHMV 1995)

(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \to \psi$ infer ψ

non-standard structures

- **Definition** : a **non-standard structure** is a 4-tuple $\mathcal{M} = (S, S', \bar{\pi}, R)$ where
  (i) S is a space of standard states,
  (ii) S' is a space of non-standard states,
  (iii) $R \subseteq S \cup S' \times S \cup S'$ is an accessibility relation,
  (iv) $\pi : Form(\mathcal{L}B(At)) \times S \rightarrow \{0, 1\}$ is a satisfaction relation **standard on** S

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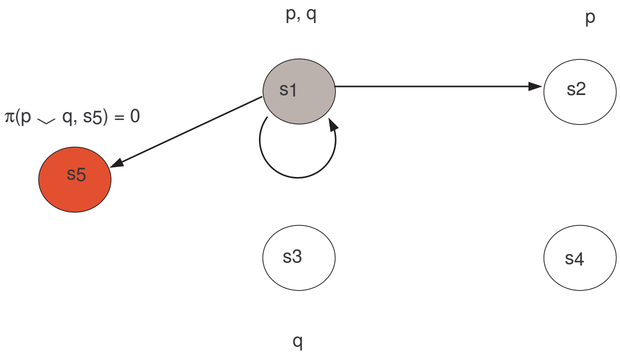
  - (iv) $\pi : Form(\mathcal{L}B(At)) \times S \rightarrow \{0, 1\}$ is a satisfaction relation **standard on** S
- **Definition** : the subjective informational content of ϕ is the set of states where ϕ is true : $[[\phi]]^*_{\mathcal{M}} = \{s \in S^* : \pi(\phi, s) = 1\}$
- Doxastic satisfaction condition : for $s \in S$, $\pi(s, B\phi) = 1$ iff for all s' s.t. sRs', $s' \in [[\phi]]^*_{\mathcal{M}}$

non-standard structures, example



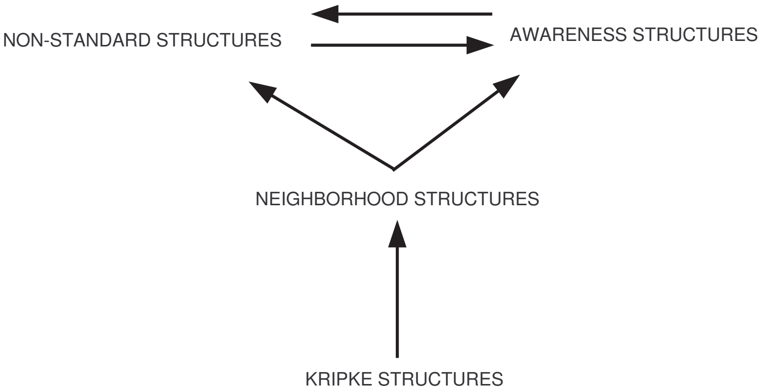
Pierre believes that p but not that $p \vee q$ since $\pi(s_5, p \vee q) = 0$ and $s_1 R s_5$.

non-standard structures, axiomatization

Minimal Epistemic Logic (Wansing 1991)

(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

respective powers



- **Definition** : let $\mathcal{L}(At)$ a propositional language ; an **implicit probabilistic structure** for $\mathcal{L}(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, P)$ where
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  (i) S is a state space,
  (ii) π is a valuation,
  (iii) P is a probability distribution on S.
- An agent believes to degree r a formula $\varphi \in Form(\mathcal{L}(At))$, $PB(\varphi) = r$, iff $P([[\varphi]]_{\mathcal{M}}) = r$

probabilistic logical omniscience

- Are there forms of LO in this probabilistic framework?
  Yes :
- **Proposition** : for all probabilistic structure $\mathcal{M}$,
  (i) **deductive monotony** : if ϕ $\mathcal{M}$-implies ψ, then $PB(\phi) \leq PB(\psi)$.
  (ii) **intensionality** : if ϕ and ψ are $\mathcal{M}$-equivalent, then $PB(\phi) = PB(\psi)$.

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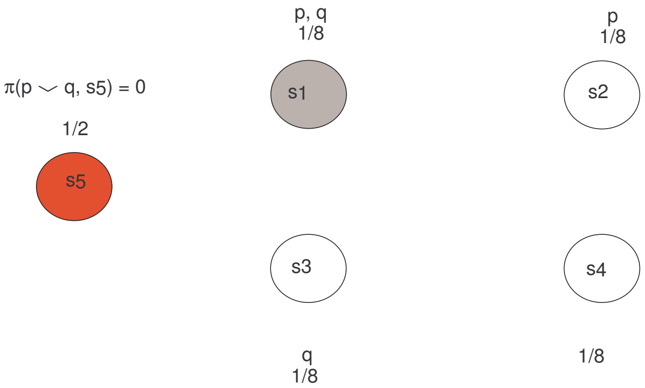
- Certainty (r = 1)
  (i) if ϕ $\mathcal{M}$-implies ψ, if it is certain that ϕ, it is certain that ψ
  (ii) if ϕ and ψ are $\mathcal{M}$-equivalent, ϕ is certain iff ψ is certain.

non-standard implicit probabilistic structures (NSIPS)

- **Definition** : let $\mathcal{L}(At)$ a propositional language ; a **non-standard implicit probabilistic structure** for $\mathcal{L}(At)$ is a 4-tuple $\mathcal{M} = (S, S', \pi, P)$ where
  (i) S is a standard state space,
  (ii) S' is a non-standard state space,
  (iii) $\pi : Form(L(At)) \times S \cup S' \rightarrow \{0, 1\}$ is a satisfaction relation which is standard on S,
  (iv) P is a probability distribution on $S^* = S \cup S'$

NSIPS, example



Pierre believes that p to degree 3/4, but that $p \vee q$ to degree 3/8.

deductive information and learning in NSIPS

- What does it mean to learn that ϕ implies ψ ?
  According to the possible-state analysis of belief : exclude states where ϕ is true but ψ false ; hence, to learn the event

$$I(\phi, \psi) = S^* - ([[\phi]]^*_{\mathcal{M}} - [[\psi]]^*_{\mathcal{M}})$$

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$$I(\phi, \psi) = S^* - ([[\phi]]^*_{\mathcal{M}} - [[\psi]]^*_{\mathcal{M}})$$

- Compatibility between conditionalization on I and deductive monotony
  Proposition : if $I(\phi, \psi)$ is learned according to Bayes rule, then deductive monotony is regained, *i.e.* $PB_I(\phi) \leq PB_I(\psi)$.

explicit probabilistic structures (EPS)

- **Definition** : the set of formulas of an explicit probabilistic language $\mathcal{L}L(At)$ based on a set At of propositional variables, $Form(\mathcal{L}L(At))$ is defined by :

$$\phi ::= p|\neg\phi|\phi \vee \psi|L_a\phi$$

where $p \in At$ and $a \in [0,1] \subseteq \mathbb{Q}$.

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- **Definition** : an **explicit probabilistic structure** for $\mathcal{L}L_a(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, P)$ where $P : S \to \Delta(S)$ assigns to every state a probability distribution on the state space.
- Satisfaction condition for L_a : $\bar{\pi}(s, L_a\phi) = 1$ iff $P(s)([[\phi]]) \geq a$

EPS, *cont.*

- Remark 1 : this language is the one proposed by Aumann 1999 and Heifetz and Mongin 2001. Fagin, Halpern and Meggido 1990 and Halpern 2003 use a different language.

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- Remark 3: from the explicit probabilistic language, one can define
  - $M_a\phi$ - the agent believes at most to degree a that ϕ - as $L_{1-a}\neg\phi$
  - $E_a\phi$ - the agent believes exactly to degree a that ϕ - as $M_a\phi \wedge L_a\phi$

EPS, axiomatization

System HM (Heifetz and Mongin 2001)

(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ
(RE) From $\phi \leftrightarrow \psi$ infer $L_a\phi \leftrightarrow L_a\psi$

(A1) $L_0\phi$
(A2) $L_a\top$
(A5)$L_a\phi \rightarrow \neg L_b\neg\phi$ $(a+b>1)$
(Def M) $M_a \leftrightarrow L_{1-a}\neg\phi$
(A8)$\neg L_a\phi \rightarrow M_a\phi$

(B) From $((\phi_1,...,\phi_m) \leftrightarrow (\psi_1,...,\psi_n))$ infer
$(\neg M_{a1}\phi_1 \wedge (\bigwedge_{i=2}^{m} L_{ai}\phi_i) \wedge (\bigwedge_{j=2}^{n} M_{bj}\psi_j) \rightarrow$
$\neg M_{(a1+...+am)-(b1+...+bn)}\psi_1)$

EPS, axiomatization, *cont.*

- some theorems :
  (RN) From $\phi \rightarrow \psi$ infer $L_a\phi \rightarrow L_a\psi$
  (A2+) $\neg L_a\bot$ $(a > 0)$
  (A3) $L_a(\phi \wedge \psi) \wedge L_b(\phi \wedge \neg\psi) \rightarrow L_{a+b}\phi$ $(a + b \leq 1)$
  (A4)$\neg L_a(\phi \wedge \psi) \wedge \neg L_b(\phi \wedge \neg\psi) \rightarrow \neg L_{a+b}\phi$ $(a + b \leq 1)$
  (A7) $L_a\phi \rightarrow L_b\phi$ $(b < a)$

non-standard explicit probabilistic structures (NSEPS)

- **Définition** : a **non-standard explicit probabilistic structure** for $\mathcal{L}L_a(At)$ is a 4-tuple $\mathcal{M} = (S, S'\pi, P)$ where π is standard on S and $P : S^* \longrightarrow \Delta(S^*)$ assigns to every state a probability distribution on the state space.

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- Remark : $\top$ and $\bot$ are added with the meanings :
  - $\top$ is what the agent recognizes as necessarily true, then for every $s \in S^*$, $\pi(s, \top) = 1$
  - $\bot$ is what the agent recognizes as necessarily false, then for no $s \in S^*$, $\pi(s, \bot) = 1$

NSEPS, axiomatization

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Completeness Theorem : $\models_{NSEPS} \phi$ iff $\vdash_{MPL} \phi$

axiomatization, *cont.*

- Proof of completeness :
  - method of canonical models with filtration
  - for each formula ϕ, one defines a subset of formulas ; in this subset of formulas, *atoms* are maximal coherent set of formulas
  - the hard stuff is to define a canonical probability distribution
  - idea : let Γ be an atom ; in the (standard state) s_Γ, $P(s_\Gamma)$ is an equiprobability on non-standard states s.t. for every formula χ s.t. some $L_a\chi$ are in Γ, a proportion b^* of non-standard states make χ true, where $b^* = \max_b L_b\chi \in \Gamma$.

Dutch Book and bayesianism

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- Bayesian answer: probabilistic beliefs are the necessary consequences of **practical rationality**
- The bayesian answer is supported by the **Dutch Book argument** (de Finetti, Ramsey): if Bob's partial beliefs are not probabilistic and are mirrored in betting quotients, then Alice can devise from these very betting quotients a set of bets such that whatever happens, Bob will lose money. (From Gillies, 2000.)

Dutch Book-coherence

- Basic Idea: Bob chooses betting quotient q_ϕ toward formula ϕ; Alice chooses stake S_ϕ (positive or negative). Bob pays $q_\phi.S_\phi$ to Alice to enter the following bet:
  - if ϕ is true, then Bob wins S_ϕ
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  - if ϕ is true, then Bob wins S_ϕ
  - if ϕ is not true, then Bob wins nothing
- **Definition**: Bob is **DB-coherent** toward a set of formulas Γ iff he chooses betting quotients such that Alice cannot assign stakes to formulas such that for every possible state $s \in S$, if s is realized Bob loses money - a Dutch Book against Bob

Example

- **Claim**
  If Bob chooses a betting quotient $q_\phi > 1$ for any formula ϕ, then Alice can devise a Dutch Book against Bob

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  If Bob chooses a betting quotient $q_\phi > 1$ for any formula ϕ, then Alice can devise a Dutch Book against Bob
  How ? It suffices for Alice to assign $S_\phi > 0$.
- By the same token, one can justify that partial beliefs should be positive, additive, etc.

imperfect logicians lose money

- **Claim 1**
  Let θ be a formula that is logically valid but that Bob doesn't recognizes as such. Suppose that Bob chooses the betting quotient $q_\theta < 1$. Then Bob is DB-incoherent.

imperfect logicians lose money

- **Claim 1**
  Let θ be a formula that is logically valid but that Bob doesn't recognizes as such. Suppose that Bob chooses the betting quotient $q_\theta < 1$. Then Bob is DB-incoherent.
- **Claim 2**
  Let ϕ, ψ two formulas such that ϕ implies ψ. Suppose that Bob chooses betting quotients $q_\phi > q_\psi$ - Bob violates (probabilistic) deductive monotony. Then Bob is DB-incoherent.
  Example:
  - Bob chooses $q_p = 3/4$ and $q_{p \vee q} = 3/8$
  - Alice chooses $S_p = 1$ and $S_{p \vee q} = -1$

logical assumptions of Dutch Books

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- If Bob is not omniscient, Bob is DB-**in**coherent
- A crucial requirement in the Dutch Book argument is that Alice doesn't have more information than Bob
- But Alice is logically omniscient is the sense that she doesn't have to take into account what happens in non-standard states
- What happens if one extends to logical domain the requirement that Alice doesn't have more information than Bob?

non-standard Dutch Books

- **Definition**: Bob is NSDB-coherent with respect to state space S^* including non-standard states toward a set of formulas Γ iff he assigns to every formula in Γ a betting quotient such that Alice cannot assign stakes to formulas such that for every possible state $s \in S^*$, if s is realized Bob loses money.

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- **Fact:** NSDB-coherence doesn't justify *full* probabilistic beliefs but justifies *minimal* probabilistic beliefs:
  - justified: for every formula ϕ, $0 \leq q_\phi \leq 1$; $q_\top = 1$
  - *not* justified: if ϕ implies ψ, $q_\phi \leq q_\psi$; if ϕ and ψ are logically incompatible, $q_{\phi \vee \psi} = q_\phi + q_\psi$

conclusion

- The shift to non-standard probabilistic structures is only a first step from the decision-theoretic point of view: one has still to "plug" the new doxastic model on an axiological model and a criterion of choice
- The next step is to provide a *representation theorem* à la Savage

conclusion, *cont.*

- **Subjective Expected Utility:** if agent's preferences conform to a set of conditions Π, then there exists a probability distribution P on S and a utility function u s.t. preferences can be represented by expected utility defined on P and u

conclusion, *cont.*

- **Subjective Expected Utility:** if agent's preferences conform to a set of conditions Π, then there exists a probability distribution P on S and a utility function u s.t. preferences can be represented by expected utility defined on P and u
- **Subjective Expected Utility without LO:** if agent's preferences conform to a set of conditions Π', then there exists non-standard states S', a probability distribution P on $S \cup S'$ and a utility function u s.t. preferences can be represented by expected utility defined on P and u
- Question: what is Π'?