CHAPTER 2

Impossible States at Work: Logical Omniscience and Rational Choice

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Abstract

Logical omniscience is a never-ending problem in epistemic logic, the main model of full beliefs. It is seldom noticed that probabilistic models of partial beliefs face the same problem. As far as choice models are built on such doxastic models, they necessarily inherit the problem as well. Following some philosophical (Hacking, 1967) and decision-theoretic (Lipman, 1999) contributions, we advocate the use of nonstandard or impossible states to tackle this issue. First, we extend the non-standard structures to the probabilistic case; an axiom system is devised, i.e. proved to be complete with respect to nonstandard probabilistic structures. Second, we show how to substitute weakened doxastic models for the idealized ones in choice models, and discuss the questions raised by this "unidealization".

Keywords: Bounded rationality, epistemic logic, logical omniscience, probabilistic logic.

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1. Introduction

Let us imagine an agent that could solve any stochastic decision process, whatever the number of periods, states and alternatives may be; that could find a Nash equilibrium in any finite game, whatever the number of players and strategies

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may be; more generally that would have a perfect mathematical knowledge and, still more generally, which would know all the logical consequences of his or her beliefs. By definition, this agent would be described as *logically omniscient*.

For sure, logical omniscience is an highly unrealistic hypothesis from the psychological point of view. Yet, this is the cognitive situation of agents in the main current doxastic models, i.e. models of beliefs. The issue has been raised a long time ago in epistemic logic (Hintikka, 1975, see the recent survey in Fagin *et al.*, 1995), which is the classical model of *full beliefs*. In particular, it has been recognized that logical omniscience is one of the most uneliminable cognitive idealizations, because it is an immediate consequence of the core principle of the modeling: the representation of beliefs by a space of possible states.

What is the relevance for rational choice theory? A standard decision model has three fundamental building blocks:

1. a model of beliefs, or doxastic model,
2. a model of desires, or axiological model, and
3. a criterion of choice, which, given beliefs and desires, selects the "appropriate" actions.

In choice under uncertainty, the classical model assumes that the doxastic model is a probability distribution on a state space, the axiological model a utility function on a set of consequences and the criterion is the maximization of expected utility. In this case, the doxastic model is a model of *partial beliefs*. But there are choice models which are built on a model of full beliefs: this is the case of models like maximax or minimax (Luce and Raiffa, 1985, Chapter 13), where one assumes that the agent takes into account the subset of possible states that is compatible with his or her beliefs.

The point is that, in both cases, *the choice model inherits the cognitive idealizations of the doxastic model*. Consequently, the choice model is cognitively *at least as unrealistic as* the doxastic model upon which it is based. Indeed, a choice model is strictly more unrealistic than its doxastic model since it assumes furthermore the axiological model and the implementation of the choice criterion. Hence, one of the main sources of cognitive idealization in choice models is the logical omniscience of their doxastic model; the weakening of logical omniscience in a decision-theoretic context is therefore one of the main ways to build more realistic choice models, i.e. to achieve bounded rationality.

Surprisingly, whereas there has been extensive work on logical omniscience in epistemic logic, there has been very few attempts to investigate the extension of the putative solutions to the probabilistic representation of beliefs (*probabilistic case*) and to models of decision making (*decision-theoretic case*).[1]

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[1] One important exception is Lipman (1999).

The aim of this paper is to make some progress in filling this gap. Our method is the following one: given that a huge number of (putative) solutions to logical omniscience have been proposed in epistemic logic, we will not start from scratch, but we will consider extensions of the main current solutions. Our main claim is that the solution that we will call the "nonstandard structures" constitute the best candidate to this extension.

The remainder of the paper proceeds as follows. In Section 2 the problem of logical omniscience and its most popular solutions are briefly recalled. Then, it shall be argued that, among these solutions, nonstandard structures are the best basis for an extension to probabilistic and decision-theoretic cases. Section 3 is devoted to the probabilistic case and states our main result: an axiomatization for nonstandard explicit probabilistic structures. In Section 4, we discuss the extension to the decision-theoretic case.

2. Logical omniscience in epistemic logic

2.1. Epistemic logic

Problems and propositions related to logical omniscience are best expressed in a logical framework, usually called "epistemic logic" (see Fagin *et al*. (1995) for an extensive technical survey and Stalnaker (1991), reprinted in Stalnaker (1998) for an illuminating philosophical discussion), which is nothing but a particular interpretation of modal logic. Here is a brief review of the classical model: Kripke structures.

First, we have to define the *language* of propositional epistemic logic. The only difference with the language of propositional logic is that this language contains a doxastic operator B: $B\phi$ is intended to mean "the agent believes that ϕ".

Definition 1. *The set of formulas of an epistemic propositional language* $\mathscr{L}B(At)$ *based on a set* At *of propositional variables Form* ($\mathscr{L}B(At)$), *is defined by* [2]

$$\phi ::= p|\neg\phi|\phi \vee \psi|\phi \wedge \psi|B\phi$$

The interpretation of the formulas is given by the famous Kripke structures.

Definition 2. *Let* $\mathscr{L}B(At)$ *an epistemic propositional language*; *a* ***Kripke structure*** *for* $\mathscr{L}B(At)$ *is a 3-tuple* $\mathscr{M} = (S, \pi, R)$ *where*

(i) S *is a state space,*
(ii) π: $At \times S \rightarrow \{0, 1\}$ *is a valuation, and*
(iii) $R \subseteq S \times S$ *is an accessibility relation.*

[2] This formulation (the so called Backus-Naur Form) means, for instance, which the propositional are formulas, that if ψ is a formula, $\neg\psi$ too, and so on.

Intuitively, the accessibility relation associates to every state the states that the agent considers possible given his or her beliefs. π associates to every atomic formula, in every state, a truth value; it is extended in a canonical way to every formula by the satisfaction relation.

Definition 3. *The* ***satisfaction relation****, labelled* $\vDash$ *extends* π *to every formula of the language according to the following conditions*:

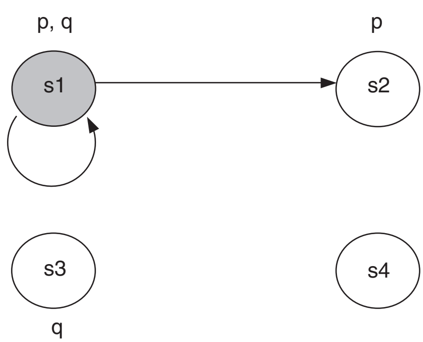
(i) $\mathcal{M}, s \vDash p$ *iff* $\pi(p; s) = 1$,
(ii) $\mathcal{M}, s \vDash \phi \wedge \psi$ *iff* $\mathcal{M}, s \vDash \phi$ *and* $\mathcal{M}, s \vDash \psi$,
(iii) $\mathcal{M}, s \vDash \phi \vee \psi$ *iff* $\mathcal{M}, s \vDash \phi$ *or* $\mathcal{M}, s \vDash \psi$,
(iv) $\mathcal{M}, s \vDash \neg\phi$ *iff* $\mathcal{M}, s \neq \phi$, and
(v) $\mathcal{M}, s \vDash B\phi$ *iff* $\forall s' s.t.\ sRs', \mathcal{M}, s' \vDash \phi$.

The specific doxastic condition contains what might be called the **possible-state analysis of belief**. It means that an agent believes that ϕ if, in all the states that (according to him or her) could be the actual state, ϕ is true: *to believe something is to exclude that it could be false*. Conversely, an agent does not believe ϕ if, in some of the states that could be the actual state, ϕ is false: *not to believe is to consider that it could be false*. This principle will be significant in the discussions below.

Example 1. $S = \{s_1, s_2, s_3, s_4\}$; *p* (“*it's sunny*”) *is true in* s_1 *and* s_2, *q* (“*it's windy*”) *in* s_1 *and* s_4. *Suppose that* s_1 *is the actual state and that in this state the agent believes that p is true but does not know if q is true*. Figure 1 represents this situation, omitting the accessibility relation in the non-actual states.

Definition 4. *Let* $\mathcal{M}$ *be a Kripke structure*; *in* $\mathcal{M}$, *the set of states where* ϕ *is true, or the* **proposition** *expressed by* ϕ, *or the* ***informational content*** *of* ϕ, *is noted* $[[\varphi]]_{\mathcal{M}}\{s : \mathcal{M}, s \vDash \varphi\}$.

Figure 1. Kripke structures



To formulate logical omniscience, we need lastly to define the following semantical relations between formulas.

Definition 5. ϕ *$\mathcal{M}$-**implies*** ψ *if* $[[\phi]]_{\mathcal{M}} \subseteq [[\psi]]_{\mathcal{M}}$.$\phi$ *and* ψ *are* ***$\mathcal{M}$-equivalent*** if $[[\phi]]_{\mathcal{M}} = [[\psi]]_{\mathcal{M}}$.

There are several forms of logical omniscience (see Fagin *et al.*, 1995); the next proposition shows that two of them, deductive monotony and intensionality, hold in Kripke structures:

Proposition 1. *Let $\mathcal{M}$ be a Kripke structure and $\phi, \psi \in \mathcal{L}B(At)$.*

(i) ***Deductive monotony**: if ϕ $\mathcal{M}$-implies ψ*, then *$B\phi$ $\mathcal{M}$-implies $B\psi$ and*
(ii) ***Intensionality**: if ϕ* and *ψ* are *$\mathcal{M}$-equivalent, then $B\phi$ and $B\psi$ are $\mathcal{M}$-equivalent.*

Both properties are obvious theorems in the axiom system *K*, which is sound and complete for Kripke structures:

System K

(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ
(K) $B\phi \wedge B(\phi \rightarrow \psi) \rightarrow B\psi$
(RN) From ϕ, infer $B\phi$

2.2. *Three solutions to logical omniscience*

A huge number of solutions have been proposed to weaken logical omniscience, and arguably no consensus has been reached (see Fagin *et al.*, 1995).[3] We identify three main solutions to logical omniscience, which are our three candidates to an extension to the probabilistic or decision-theoretic case. There is probably some arbitrariness in this selection, but they are among the most used, natural and powerful existing solutions.

2.2.1. Neighborhood structures

The "neighborhood structures", sometimes called "Montague–Scott structures" are our first candidate. The basic idea is to make explicit the *propositions* that the agent believes; the neighborhood system of an agent at a given state is precisely the set of propositions that the agent believes.

[3] We have contributed to this industry by defending the use of substructural logics in Cozic (2006); this setting is not tractable enough for the aim of the current paper.

Definition 6. *A* ***neighborhood structure*** *is a 3-tuple* $\mathscr{M} = (S, \pi, V)$ *where*

(i) *S is a state space,*
(ii) $\pi: At \times S \to \{0, 1\}$ *is a valuation, and*
(iii) $V: S \to \wp((\wp(S))$, *called the agent's* ***neighborhood system****, associates to every state a set of propositions.*

The conditions on the satisfaction relation are the same, except for the doxastic operator:

$$\mathscr{M}, s \vDash B\phi \text{ iff } [[\phi]]_{\mathscr{M}} \in V(s)$$

It is easy to check that deductive monotony is invalidated by neighborhood structures, as shown by the following example.

Example 2. *Let us consider the first example and replace the accessibility relation by a neighborhood system;* $V(s_1)$ *contains* $\{s_1, s_2\}$ *but not* $\{s_1, s_2, s_3\}$*. Then, in* s_1*,* Bp *is true but not* $B(p \vee q)$*. This is represented in* Figure 2.[4]

As expected, one can regain deductive monotony by closing the neighborhood systems under supersets. Nonetheless, the axiomatization presented below[5] makes clear that the power of neighborhood system is limited: intensionality cannot be weakened.

System E (Chellas, 1980)

(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \to \psi$ infer ψ
(RE) From $\phi \leftrightarrow \psi$ infer $B\phi \leftrightarrow B\psi$

2.2.2. Awareness structures

The second solution, due to R. Fagin and J. Halpern (1988),[6] are the "awareness structures". The basic idea is to put a *syntactical filter* on the agent's beliefs. The term "awareness" suggests that this can be interpreted as reflecting the agent's awareness state, but other interpretations are conceivable as well.

Definition 7. *An* ***awareness structure*** *is a 4-tuple* (S, π, R, A) *where*

(i) *S is a state space,*
(ii) $\pi: At \times S \to \{0, 1\}$ i*s a valuation,*

[4] This reccuring example is not chosen for its cognitive realism, but because it makes the comparison of different solutions easy.
[5] The system E is strong and complete with respect to neighborhoods structures.
[6] What we call "awareness structures" is called in the original paper "logic of general awareness".

Figure 2. Neighborhood structures

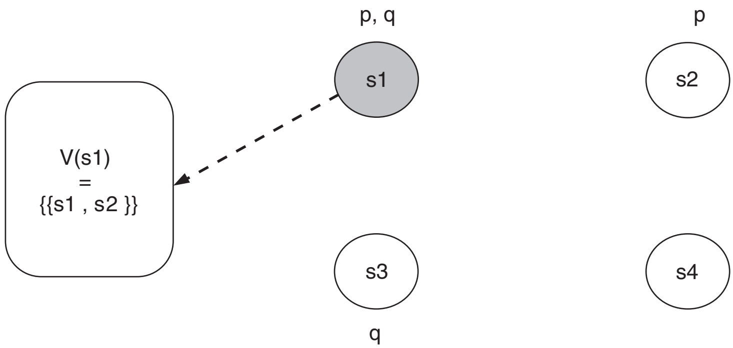
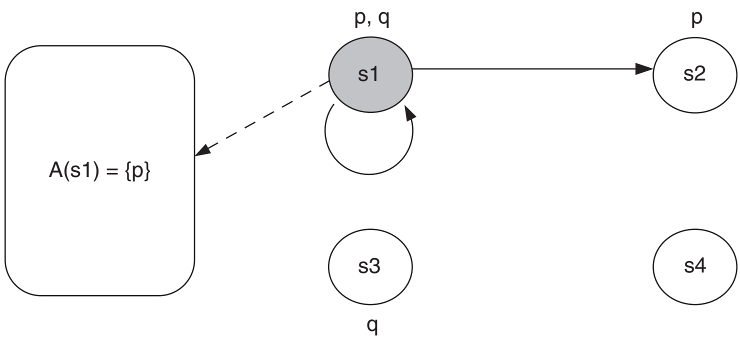


Figure 3. Awareness structures



(iii) *$R \subseteq S \times S$ is an accessibility relation, and*
(iv) *A:$S \rightarrow$ Form ($\mathscr{L}B(At)$) is a function which maps every state in a set of formulas* (“*awareness set*”).

The new condition on the satisfaction relation is the following:

$\mathscr{M}, s \vDash B\phi$ *iff* $\forall s'$ *s.t.* $sRs' \in [[\phi]]\mathscr{M}$ and $\phi \in A(s)$

This new doxastic condition permits to weaken *any form* of logical omniscience; in particular, our example shows how to model an agent who violates deductive monotony.

Example 3. *Let us consider our example and stipulate that $A(s_1) = \{p\}$. Then it is still the case that Bp is true in s_1, but not $B(p \vee q)$. This is represented in* Figure 3.

If one keeps the basic language $\mathscr{L}B(At)$, one obtains as axiom system a minimal epistemic logic which eliminates any form of logical omniscience:

Minimal Epistemic Logic (FHMV, 1995)

(PROP) Instances of propositional tautologies
(MP) From $\phi$and $\phi \rightarrow \psi$ infer ψ

2.2.3. Nonstandard structures

We now switch to our last solution: the nonstandard structures, which are sometimes called "Kripke structures with impossible states". Contrary to the two preceding solutions, neither the accessibility relation nor the doxastic condition are modified. What is revised is the underlying state space or, more precisely, the nature of the satisfaction relation in certain states of the state space.

Definition 8. *A **nonstandard structure** is a 5-tuple* $\mathscr{M} = (S, S', \pi, R, \vDash)$ *where*

(i) *S is a space of standard states,*
(ii) *S' is a space of nonstandard states,*
(iii) *$R \subseteq S \cup S' \times S \cup S'$ is an accessibility relation,*
(iv) π :Form $(\mathscr{L}B(At)) \times S \rightarrow \{0, 1\}$ *is a valuation on S, and*
(v) $\vDash$ *is a satisfaction relation which is standard on S (recursively defined as usual) but arbitrary on S'.*

In nonstandard structures, there are no *a priori* constraints on the satisfaction relation in nonstandard states. For instance, in a nonstandard state s', both ϕ and $\neg\phi$ can be false. For every formula ϕ, one might therefore distinguish its *objective informational content* $[[\phi]]_{\mathscr{M}} = \{s \in S : \mathscr{M}, s \vDash \phi\}$ from its *subjective informational content* $[[\varphi]]^{*}_{\mathscr{M}} = \{s \in S^{*} = S \cup S' : \mathscr{M}, s \vDash \phi\}$. In spite of appearances, this generalization of Kripke structures is arguably natural as soon as one accepts the possible-state analysis of beliefs. Recall that, according to this analysis,

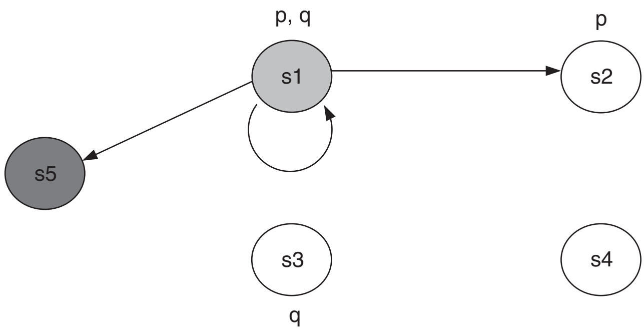
- to believe that ϕ is to exclude that ϕ could be false and
- not to believe that ψ is not to exclude that ψ could be false.

In consequence, according to the possible-state analysis, to believe that ϕ but not to believe one of its logical consequences ψ is to consider as possible at least one state where ϕ is true but ψ false. By definition, a state of this kind is logically nonstandard. Nonstandard structures is the most straightforward way to keep the possible-state analysis of beliefs.[7]

Example 4. *Let us consider our example but add a nonstandard state in* $S' = \{s_5\}$*; we stipulate that* $\mathscr{M}, s_5 \vDash p$*, but that* $\mathscr{M}, s_5 \vDash (p \vee q)$*. Then in s_1, Bp is true but not* $B(p \vee q)$*. This is represented in* Figure 4.

[7] For a more extensive defense of the solution, see Hintikka (1975) or, more recently, Barwise (1997).

Figure 4. Nonstandard structures



3. The probabilistic case

Mainstream decision theory is based on doxatic models of *partial beliefs*, not of full beliefs. Hence weakenings of logical omniscience in the framework of doxastic logic does not give directly a way to weaken logical omniscience that is appropriate for decision theorists. The aim of this section is to study the probabilistic extension of doxastic models without logical omniscience.

3.1. Probabilistic counterpart of logical omniscience

First, we have to define the probabilistic counterparts of logical omniscience. In the usual (non-logical) framework, if P is a probability distribution on S,[8] then the following property is the counterpart of logical omniscience: if $E \subseteq E'$, then $P(E) \leq P(E')$.

But to be closer to the preceding section, it is better to work with an elementary [9] logical version of the usual probabilistic model:

Definition 9. *Let $\mathscr{L}(At)$ a propositional language; a* ***probabilistic structure***[10] *for $\mathscr{L}(At)$ is a 3-tuple $\mathscr{M} = (S, \pi, p)$ where*

(i) *S is a state space,*
(ii) *π is a valuation, and*
(iii) *P is a probability distribution on S.*

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[8] In the paper, to avoid complications that are unnecessary for our purpose, we suppose that S is finite and that P is defined on $\wp(S)$.
[9] "Elementary" because there is no doxastic operator in the object-language.
[10] See Fagin and Halpern, (1991). For a recent reference on logical formalization of probabilistic reasoning, see Halpern (2003).

We will say that an agent believes to degree r a formula $\phi \in Form(\mathscr{L}(At))$, symbolized by $CP(\phi) = r$, if $P([[\phi]]_{\mathscr{M}}) = r$.[11] We can state the precise probabilistic counterparts of logical omniscience:

Proposition 2. The following holds in probabilistic structures:

(i) *deductive monotony: if ϕ $\mathscr{M}$-implies ψ, then $CP(\phi) \leq CP(\psi)$ and*
(ii) *intensionality: if ϕ and ψ are $\mathscr{M}$-equivalent, then $CP(\phi) = CP(\psi)$.*

One can check that these are indeed the *counterparts* of logical omniscience by looking at the limit case of *certainty*, i.e. of maximal degree of belief: (i) if an agent is certain that ϕ and if ϕ $\mathscr{M}$-implies ψ, then the agent is certain that ψ as well; (ii) if ϕ and ψ are $\mathscr{M}$-equivalent, then an agent is certain that ϕ iff he or she is certain that ψ.

Which of the three solutions to choose for this extension?

(a) First, we should eliminate neighbordhood structures because their power is limited: intensionality is a too strong idealization. This is especially sensitive in a decision context, where, under the label of "framing effects", it has been recognized for a long time that logically equivalent formulations of a decision problem could lead to different behaviors.

(b) Second, the extension of awareness structures seems intrisically tricky. Suppose that an agent believes ϕ to degree r_ϕ and ψ to degree $r\psi$ with ϕ $\mathscr{M}$-implying ψ and $r_\phi > r\psi$. This is a failure of deductive monotony. Now, in an analogous situation, the way awareness structures proceed in epistemic logic is by "dropping" the formula ψ. Let us apply this method to the probabilistic case: we would say that an agent believes that ϕ to degree r if $P([[\phi]])\mathscr{M}) = r$ *and* he or she is aware of ϕ. But no one could model a situation like the preceding one: either the agent is aware of ψ and in this case necessarily he or she believes that ψ to a degree $r\psi \geq r_\phi$; or he or she is not aware of ψ, and it this case he or she has no degree of belief toward ψ. This is not a knock-down argument, but it implies that if one wants to extend awareness structures, one has to make it substantially more sophisticated.

(c) Lastly, the extension of awareness structure is problematic in our perspective, i.e. a perspective of decision-theoretic application. To see why, let us notice that a criterion choice like expected utility might be seen as a function whose first argument is a doxastic model and second argument an axiological model. If we would extend the awareness structures, the first value of an expected utility criterion would not be any more a simple probability distribution. Consequently, *we should have to revise our choice criterion*. For sure, nothing precludes such a move, but simplicity recommends another tactic.

We are therefore left with nonstandard structures. Nonstandard structures do not suffer from the above-mentioned troubles: they are as powerful as one

[11] Note that $CP(\phi) = r$ is in the meta-language, not in the object-language.

can wish, the extension is intrinsically simple and they should permit to keep usual choice criterion when embedded in a choice model. This is our motivation, but now we have to turn to positive arguments.[12]

3.2. Nonstandard implicit probabilistic structures

To give the basic insights and show the fruitfulness of the proposition, we will continue to work in the elementary setting where no doxastic operators are in the object-language.

Definition 10. *Let $\mathscr{L}(At)$ a propositional language;* a ***non-standard implicit probabilistic structure*** for $\mathscr{L}(At)$ *is a 5-tuple* $\mathscr{M} = (S, S', \pi \vDash, P)$ *where*

(i) *S is a standard state space,*
(ii) *S' is a nonstandard space,*
(iii) *$\pi : Form\ (L(At)) \times S \rightarrow \{0,\ 1\}$ is a valuation on S,*
(iv) *$\vDash$ is a satisfaction relation which is standard on S but arbitrary on S', and*
(v) *P is a probability distribution on $S^* = S \cup S'$.*

As in the set-theoretic case, one can distinguish the objective informational content of a formula, i.e. the standard states where this formula is true, and the subjective informational content of a formula, i.e. the states where this formula is true.

To obtain the expected benefit, the nonstandard probabilistic structures should characterize the agent's doxastic state on the basis of *subjective* informational content: an agent believes a formula ϕ to degree r, $CP(\phi) \vDash r$, if $P([[\phi]]^*_{\mathscr{M}}) = r$. It is easy to check that, in this case, logical omniscience can be utterly controlled.

Example 5. *Let us take the same space state as in the preceding examples. Suppose that the agent has the following partial beliefs*: $CP(p) > CP(p \vee q)$. *This can be modeled in the following way:* $S' = \{s_5\}$, $s_5 \in [[p]]^*_{\mathscr{M}}$ *but* $s_5 \notin [[p \vee q]]^* \mathscr{M}$, $P(s_1) = P(s_2) = P(s_3) = P(s_4) = 1/8$ and $P(s_5) = 1/2$. This is represented in Figure 5.

3.3. Special topics: deductive information and additivity

This extension of nonstandard structures is admittedly straightforward and simple. It gives immediately the means to weaken logical idealizations. Furthermore, it opens perspectives specific to the probabilistic case; two of them will be briefly mentioned.

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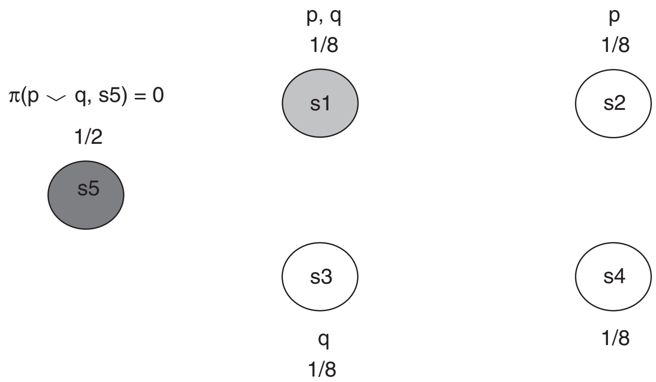
[12] A similar idea has been defended a long time ago by I. Hacking who talks about "personal possibility", by contrast with "logical possibility". We will not develop the point here, but this contribution can be seen as a formalization of Hacking's insights (Hacking, 1967).

Figure 5. Probabilistic non-standard structures



Deductive information and learning. First, one can model the fact that an agent acquires not only empirical information but *deductive information*; in nonstandard structures, this corresponds to the fact that *the agent eliminates nonstandard states*.

Let us come back to our generic situation. Suppose that our agent learns that ϕ implies ψ. This means that he or she learns that the states where ϕ is true but ψ false are impossible. This is equivalent to say that he or she learns the event

$$\mathrm{I} = S^* - ([[\phi]]^*_{\mathcal{M}} - [[\psi]]^*_{\mathcal{M}})$$

To be satisfying, such a notion of deductive information must respect a requirement of compatibility between revising and logical monotony: if the agent learns that ϕ implies ψ and revise his or her beliefs upon this fact, his or her new probability distribution should conform to logical monotony with respect to ϕ and ψ. One can check that it is the case with the main revising rule, i.e. conditionalization.

Proposition 3. *If I is learned following the conditionalization, then deductive monotony is regained, i.e.* $CP_I(\phi) \leq CP_I(\psi)$.

Example 6. *This can be checked in the preceding example:* $I = S = \{s_1, s_2, s_3, s_4\}$. *By conditionalization,* $CP_I(p) = 1/2$ *whereas* $CP_I(p \vee q) = 3/4$.

Additivity. A second topic is additivity. From a logical point of view, one can define additivity as follows:

Definition 11. *$\mathcal{M}$ is (logically)* ***additive*** *if, when ϕ and ψ are logically incompatible,* $CP(\phi)+CP(\psi) = CP(\phi \vee \psi)$.

Additivity is of course the core of the probabilistic representation of beliefs, and alternative representations of beliefs depart often from probability on this point. For example, in the Dempster–Shafer theory (Shafer, 1976), the so-called belief function is superadditive (in our notation, $CP \vee (\phi \vee \psi) \geq CP(\phi) + CP(\psi)$) whereas its dual, the plausibility function, is subadditive $CP(\phi \vee \psi) \leq CP(\phi) + CP(\psi)$).

A noteworthy aspect of probabilistic nonstandard structures is that the freedom of the connectives' behavior in nonstandard states permits us to have a very flexible framework with respect to additivity: simple conditions on the connectives imply general properties concerning additivity.

Definition 12. *Let $\mathcal{M} = (S, S', \pi, \models, P)$ a nonstandard probabilistic structure; $\mathcal{M}$ is* $\vee$ ***-standard*** *if for every formulas $\phi, \psi, [[\phi \vee \psi]]^*_{\mathcal{M}} = [[\phi]]^*_{\mathcal{M}} \cup [[\psi]]^*_{\mathcal{M}}$.*

This means that the disjunction behaves in the usual way in nonstandard states; a trivial consequence of this is that the structure $\mathcal{M}$ is (logically) subadditive.

Proposition 4. *If $\mathcal{M}$ is $\vee$-standard, then it is (logically) subadditive.*

To be a little bit more general, one can consider the (logical) inclusion–exclusion rule:

$$CP(\phi \vee \psi) = CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$$

One can define (logical) **submodularity** (respectively supermodularity or convexity) as: $CP(\phi \vee \psi) \leq CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$ (respectively $CP(\phi \vee \psi) \geq CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$). It is clear that to control submodularity, we have to control the conjunction's behavior.

Definition 13. *Let $\mathcal{M} = (S, S', \pi, \models, P)$ a probabilistic nonstandard structure;*

(i) *$\mathcal{M}$ is **negatively** $\wedge$**-standard** if for every formulas ϕ, ψ, when , $\mathcal{M}$, $s \not\models \phi$, or $\mathcal{M}, s \not\models \psi$, then $\mathcal{M}, s \not\models \phi \wedge \psi$.*
(ii) *$\mathcal{M}$ is **positively** $\wedge$**-standard** if for every formulas ϕ, ψ, when $\mathcal{M}, s \models \phi$, or $\mathcal{M}, s \models \psi$, then, $\mathcal{M}, s \models \phi \wedge \psi$.*

Proposition 5. *Suppose that $\mathcal{M}$ is $\vee$-standard;*

– *if $\mathcal{M}$ is negatively $\wedge$-standard, then submodularity holds*
– *if $\mathcal{M}$ is positively $\wedge$-standard, then supermodularity holds*

Proof: see the Appendix.

3.4. Nonstandard explicit probabilistic structures

Implicit probabilistic structures are not very expressive; to have a true analogon of epistemic logic, we have to start from an object-language that contains (partial) doxastic operator.

Following Aumann (1999) and Heifetz and Mongin (2001), we consider the operator L_a.[13,14] The intuitive meaning of $L_a\phi$ is: the agent believes at least to degree a that ϕ. Note that we add the usual symbols $\top, \bot$: $\top$ is what the agent recognizes as necessarily true and $\bot$ is what he or she recognizes as necessarily false.

Definition 14. *The set of formulas of an explicit probabilistic language $\mathscr{L}L(At)$ based on a set At of propositional variables, Form $(\mathscr{L}L(At))$ is defined by:*

$$\phi ::= p| \perp |\top|\neg\phi|\phi \vee \psi|L_a\phi$$

where $P \in At$ and $a \in [0,1] \subseteq \mathbb{Q}$.

The corresponding structures are an obvious extension of implicit nonstandard structures.

Definition 15. *A* ***nonstandard explicit probabilistic structure*** *for $\mathscr{L}L_a(At)$ is a 5-tuple $\mathscr{M} = (S, S', \pi, \vDash, P)$ where*

(i) $\vDash$ *is a satisfaction relation s.t.*
  (a) $\vDash$ *is standard on S for all propositional connectives,*
  (b) $\forall s \in S,\ \mathscr{M},\ s \vDash\ \mathscr{L}_a\ \phi$ *iff* $P(s)([[\phi]]^*_{\mathscr{M}}) \geq a$, and
  (c) $\forall s \in S \cup S', \mathscr{M}, s \vDash \top$ and $\mathscr{M}$, s$|\neq \bot$.

(ii) $P{:}S^* \to \Delta(S^*)$ *assigns to every state a probability distribution on the state space.*

In Aumann (1999), R. Aumann has failed to axiomatize (standard) explicit probabilistic structures, but Heifetz and Mongin (2001) have recently devised an axiom system that is (weakly) complete for these structures. In comparison with epistemic logic, one of the problems is that the adaptation of the usual proof method, i.e. the method of canonical models, is not trivial. More precisely, in the epistemic logic's case, it is easy to define a canonical accessibility relation on the canonical state space. This is not case in the probabilistic framework, where strong axioms are needed to guarantee that. Fortunately, the nonstandard structures permit huge simplifications, and one can devise an axiom system that essentially mimics the Minimal Epistemic Logic above described.

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[13] Economists are leading contributors to the study of explicit probabilistic structures because they correspond to the so-called *type spaces* used in games of incomplete information, in the same way that Kripke structures (with R as an equivalence relation). See Aumann and Heifetz (2002).

[14] Note that another language is used by J. Halpern in Fagin and Halpern (1991) or Halpern (2003).

Minimal Probabilistic Logic

(PROP) Instances of propositional tautologies
(MP) From $\phi$and $\phi \rightarrow \psi$ infer ψ

(A1) $\mathscr{L}_{\mathrm{a}}\phi$
(A2) $\mathscr{L}_{\mathrm{a}}\top$
(A2+) $\neg \mathscr{L}_{a} \perp (a > 0)$
(A7) $\mathscr{L}_{a}\phi \rightarrow \mathscr{L}_{b}\phi \; (b < a)$

The axioms' notation follows Heifetz and Mongin (2001) to facilitate comparison. Axioms (A2) and (A2+) reflect our semantic for $\top$ and $\perp$: the agent believes to maximal degree what he or she considers as necessarily true and does not believe to any degree what he or she considers as necessarily false. (A1) and (A7) reflect principles specific to the probabilistic case. Note that both bear on a single embedded formula ϕ: there is no doxastic reflection of a logical relation. They express something like a minimal metric of partial beliefs.

If $\vDash_{NSEPS} \phi$ means that ϕ is true in every nonstandard explicit probabilistic structure and $\vdash_{MPL} \phi$ that ϕ is provable in the Minimal Probabilistic Logic, then we are ready to state our main result:

Theorem 1. (Soundness and Completeness of MPL)

$$\vDash_{\mathrm{NSEPS}} \phi \text{ iff } \vdash_{MPL}\phi$$

Proof: see the Appendix.

4. *Insights into the decision-theoretic case*

We would like to end this paper by showing how to build choice models without logical omniscience, and which are the challenges raised by such a project.

4.1. *Choice models without logical omniscience*

The basic method to build a choice model without logical omniscience is to *substitute* one of our nonstandard structures to the original doxastic model in the target choice model. We will now show how this could be done.

One might generically see models of choice under uncertainty as based on

- a state space S,
- a set A of actions,
- a consequence function $\mathfrak{C}$: $S \times A \rightarrow C$ where C is a set of consequences,
- a utility function u: $C \rightarrow \mathbb{R}$, and
- a criterion of choice.

To complete the choice model, one adds a distribution P on S for models of choice under probabilistic uncertainty, and a set $K \subseteq S$ of states compatible with the agent's beliefs under set-theoretic uncertainty.

To rigourously extend nonstandard structures to choice models, one should translate the above described notions in a logical setting. But to give some insights, we will, on the contrary, import nonstandard structures in the syntax-free framework of conventional decision theory. Let us have a look at the following, admittedly particular, target situation: an agent knows abstractly the consequence function C, but, because of limited computational capacities, he or she is not able, at the moment of choice, to perfectly infer from the choice function the consequence of each action at each possible state. One can think about a classic two-state example of insurance application.[15] The consequence function is

$$\mathfrak{C}(s_1, x) = w - \pi x$$

$$\mathfrak{C}(s_2, x) = y + x,$$

where x, the choice variable, is the amount of money spent in insurance, s_1 the state without disaster, w the wealth in s_1, s_2 the state with a disaster and y the subsequent wealth, and π the rate of exchange. In this case, a nonlogically omniscient agent with respect to the consequence function would be such that he or she ignores the value of $\mathfrak{C}$ for some arguments.

A simple way to model this target situation would be the following one. Let us consider *extended states* w, which are composed of a (primitive) state s and a local consequence function $\mathfrak{C}_w : A \to C : w = (s,\ \mathfrak{C}_w)$. The set of extended states is intended to represent the beliefs of the agent, including his or her logically imperfect beliefs. An extended state is standard if its local consequence function is conform to the (true) consequence function: $\mathfrak{C}_w(a) = \mathfrak{C}(a, s)$; if not, it is nonstandard.

For instance, a logically imperfect agent could not know what is the consequence of action a in state s, thinking that it is possible that this consequence is c_i (let us say, the true one) or c_j. This situation would be modeled by building (at least) two extended states:

$$w_i = (s, \mathfrak{C}_{wi}) \text{ where } \mathfrak{C}_{wi}(a) = c_i \text{ and}$$

$$w_j = (s, \mathfrak{C}_{wj}) \text{ where } \mathfrak{C}_{wj}(a) = c_j.$$

A perfect logician would not have considered a possible state like w_j. On this basis, one can build choice models without the assumption of logical omniscience:

– in the case of choice under set-theoretic uncertainty, if one takes the maximin criterion, for a belief set $K \subseteq W$, the solution is

[15] From Lippman and McCall (1981).

$$Sol(A, S, W, C, \mathfrak{C}, E, u, K) = \arg\max_{a \in A} \min_{w \in K} u(\mathfrak{C}_w(a)), \text{and}$$

– in the case of choice under probabilistic uncertainty, if one takes the maximization of expected utility criterion, for a probability distribution P on W, the solution is

$$Sol(A, S, W, C, \mathfrak{C}, E, u, P) = \arg\max_{a \in A} \sum_{w \in W} P(w).u(\mathfrak{C}_w(a)).$$

4.2. Open questions

From the decision theorists point of view, the substitution we have just described is only a first step. Two fundamental questions remain.

(a) First, there is the question of the axiomatization of the new choice models, i.e. closely linked with the behavioral implications of choice models without logical omniscience. In a recent paper, B. Lipman (Lipman 1999) has remarkably tackled this issue, advocating a very similar approach. But the choice model he uses is quite specific (conditional expected utility), and one would like to compare choice models based on nonstandard structures with the savagean benchmark.

More precisely, one would like to obtain a *representation theorem* à la Savage: define conditions on a preference relation $\succcurlyeq$ such that there exists (1) a space of extended states W, (2) a probability distribution P on W and (3) a utility function u such that the preference relation could be rationalized by the expected utility defined over preceding notions.

(b) Second, the nonstandard choice models weaken only the cognitive assumptions of the (underlying) doxastic model. But there remains cognitive assumptions concerning the utility function and the choice criterion. In the approach we just described, we still assume that the agent is able to assign a precise utility to each consequence $c \in C$ and to calculate the solution to its choice criterion. Therefore, from the point of view of the bounded rationality program, our proposition is strongly incomplete.

5. Conclusion

This paper has advocated the use of nonstandard or impossible states as a general framework to "unidealize" belief and choice models. This admittedly does not permit a complete treatment of the idealizations underlying conventional choice models, but can be seen as a first step toward a fine-grained modeling of bounded rationality.

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Appendix:

Proof of Proposition 5 The proof deals only with the case of submodularity; the other is symmetric. If $[[\phi]]^*$ and $[[\psi]]^*$ are disjoint, then by hypothesis $[[\phi \wedge \varphi]* = \emptyset]]$. Therefore $CP(\phi \vee \psi) = CP(\phi) + CP(\psi) - CP(\phi \wedge \psi)$.

It follows from the definition that if$\mathscr{M},\ s \vDash \psi \wedge \phi$, then $\mathscr{M},\ s \vDash \psi$ and $\mathscr{M},\ s \vDash \phi$ (the converse does not hold). In other words,

(1) $[[\phi \wedge \psi]]^* \subseteq [[\phi]]^* \cap [[\psi]]^*$.
This implies that
(2) $P([[\phi \wedge \psi]]^*) \leq P([[\phi]]^* \cap [[\psi]]^*)$.

Since $\mathscr{M}$ is $\vee$-standard, $P([[\phi \vee \psi]]^*) = P([[\phi]]^*) + P([[\phi]]^*) - P([[\phi]]^* \cap [[\psi]]^*)$. By (2), it follows from this that

$$P([[\phi \vee \psi]]^*) \leq P([[\phi]]^*) + P([[\phi]]^*) - P([[\phi \wedge \psi]]^*).$$

■

Proof of Theorem 1

($\Rightarrow$). Soundness is easily checked and is left to the reader.

($\Leftarrow$). We have to show that $\vDash_{NSEPS} \phi$ implies $\vdash_{MPL} \phi$. First, let us notice that the Minimal Probabilistic Logic (*MPL*) is a "modal logic" (see Blackburn *et al*., 2001, p. 191): a set of formulas (1) that contains every propositional tautologies and (2) that is closed by *modus ponens* and uniform substitution. One can then apply the famous Lindenbaum Lemma.

Definition 16.

(i) *A formula ϕ is **deducible** from a set of formulas Γ, symbolized $\Gamma \vdash \phi$, if there exists some formulas $\psi 1, \ldots, \psi n$ in Γ s.t. $\vdash (\psi 1 \wedge \ldots \wedge \psi n) \rightarrow \phi$.*
(ii) *A set of formulas Γ is $\wedge$-**consistent** if it is false that $\Gamma \vdash \wedge\bot$.*
(iii) *A set of formulas Γ is **maximally** $\wedge$-**consistent** if (1) it is $\wedge$-consistent and (2) if it is not included in a $\wedge$-consi*stent set of formulas.

Lemma 1. *(Lindenbaum Lemma) If Γ is a set of $\wedge$-consistent formulas, then there exists an extension Γ^+ of Γ that is maximally $\wedge$-consistent.*

Proof: see for instance Blackburn *et al*. (2001, p. 199).

Definition 17. *Let $\phi \in L$; the language associated with ϕ, $\mathscr{L}_\phi$ is the smallest sublanguage that*

(i) *contains* ϕ*;* $\perp$ *et* $\top$,
(ii) *is closed under sub-formulas, and*
(iii) *is closed under the symbol* $\sim$ *defined as follows:* $\sim \chi := \psi$ *if* $\chi = \neg\psi$ *and* $\sim \chi := \neg\chi$ *if not.*[16]

In the language L_ϕ, one can define the analogon of the maximally $\wedge$-consistent sets.

Definition 18. *An atom is a set of formulas in* L_ϕ *which is maximally* $\wedge$*-consistent.* $At(\phi)$ *is the set of atoms.*

Lemma 2. *For every atom* Γ*,*

(i) there exists an unique extension of Γ in L, symbolized Γ^+, that is maximally $\wedge$-consistent and
(ii) $\Gamma = \Gamma^+ \cap L_\phi$.

Proof. (i) An application of Lindenbaum Lemma. (ii) Implied by the fact that Γ is maximally coherent. Suppose that there exists a formula ψ from L_ϕ in Γ^+ but not in Γ, then Γ^+would be inconsistent, i.e. excluded by hypothesis. ■

Starting from atoms, one may define the analogon of canonical structures, i.e. structures where (standard) states are sets of maximally $\wedge$-consistent formulas. In the same way, we will take as canonical standard state space the language's L_ϕ atoms.

The hard stuff is the definition of the probability distributions. The aim is to make true in s_Γ every formula $\mathscr{L}_a\chi$ in the atom Γ associated with the state s_Γ. To do that it is necessary that $P(s_\Gamma, \chi) \geq a$; this is guaranted if one takes for $P(s_\Gamma, \chi)$ the number b^* s.t. $b^* = \max\{b : \mathscr{L}_{bb}\chi \in \Gamma\}$. This can easily be done with non-standard states. It will be the case if (1) the support of $P(s_\Gamma,.)$ is included in the set of non-standard states, (2) $P(s_\Gamma,.)$ is equiprobable and (3) there is a proportion b^* of states that make χ true.

Suppose that $I\ (\Gamma)$ is the sequence of formulas in Γ that are prefixed by a doxastic operator $\mathscr{L}_a$; for every formula, one can rewrite $b^*(\chi)$ as p_i/q_i. Define $q(\Gamma) = \Pi_{i\in 1}\ qi$; $q(\Gamma)$ will be the set of nonstandard states in which $P(s_\Gamma,.)$ will be included. If the ist formula is χ, suffice it to stipulate that χ in the first $p_i \times \Pi\ q - i$ states. One may check that the proportion of states χ is true is p_i/q_i.

Definition 19. *The* ϕ***-canonical structure*** *is the structure* $\mathscr{M}_\phi(S_\phi, S', \pi_\phi, \vDash_\phi, P_\phi)$ *where*

(i) $S_\phi = \{s_\Gamma : \Gamma \in At\ (\phi)\}$,
(ii) $S'_\phi = \bigcup_{\Gamma \in At(\phi)} q(\Gamma)$,

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[16] See Blackburn et al. (2001, p. 242).

(iii) *for all standard state,* π_ϕ $(p, s_\Gamma) = 1$ *iff* $p \in s_\Gamma$,

(iv) *for all nonstandard state* $\mathscr{M}_\phi, s \models_\phi \psi$ *iff, if* $s \in q(\Gamma)$ *and* ψ *is the i-st formula prefixed by a doxastic operator in* Γ*, then s is in the* $pi \times \Pi_{q-i}$ *first states of* $q(\Gamma)$*, and*

(v) $P\phi$ $(s_\Gamma,.)$ *is an equiprobable distribution on* $q(\Gamma)$.

As expected, the ϕ-canonical structure satisfies the Truth Lemma.

Lemma 3. *(Truth Lemma) For every atom* Γ, $\mathscr{M}_\phi$, $s_\Gamma \models \psi$ *iff* $\psi \in \Gamma$.

Proof. The proof proceeds by induction on the length of the formula.

(a) $:= p$; follows directly from the definition of π_ϕ.

(b) $\psi = \psi_1 \vee \psi_2$; by definition, $\mathscr{M}_\phi, s \models_\phi \psi$ iff $\mathscr{M}_\phi, s \models_\phi \psi_1$ or $\mathscr{M}_\phi, s \models_\phi \psi_2$. Case (b) will be checked if one shows that $\psi_1 \vee \psi_2 \in \Gamma$ iff $\psi_1 \in \Gamma$ or $\psi_2 \in \Gamma$. Let us consider the extension Γ^+ of Γ; one knows that $\psi_1 \vee \psi_2 \in \Gamma^+$ iff $\psi_1 \in \Gamma^+$ or $\psi_2 \in \Gamma^+$. But $\Gamma = \Gamma^+ \cap \mathrm{L}_\phi$ and ψ_1 and $\psi_2 \in \mathrm{L}_\phi$. It follows that $s_\Gamma \models \psi_1 \vee \psi_2$ iff $\psi_1 \vee \psi_2 \in \Gamma$.

(c) $\psi = \neg\chi$. $\mathscr{M}_\phi$, $s \models \phi$ $\neg\chi$ iff $\mathscr{M}_\phi$, $s \not\models_\phi \chi$ iff (by induction hypothesis) $\chi \notin \Gamma$. Suffice it to show that $\chi \notin \Gamma$ iff $\neg\chi \in \Gamma$. (i) Let us suppose that $\chi \notin \Gamma$; χ is in L_ϕ hence, given the properties of maximally $\wedge$-consistent sets, $\neg\chi$ $\in \Gamma^+$. And since $\Gamma = \Gamma^+ \cap \mathrm{L}_\phi$, $\neg\chi$ $\in \Gamma$. (ii) Let us suppose that $\neg\chi$ $\in \Gamma$; Γ is coherent, therefore χ $\notin \Gamma$.

(d) $\psi = \mathscr{L}_a\chi$; by definition $s_\Gamma \models \mathscr{L}_a\chi$– iff $P(s_\Gamma, \chi) \geq a$. (i) Let us suppose that $P(s_\Gamma, \chi) \geq a$; then $a \leq b^*$ where $b^* = \max\{b : \mathscr{L} \in \Gamma\}$ since by definition of the canonical distribution, $P(s_\Gamma, \chi) \geq b^*$. Now, let us consider the extension Γ^+: clearly, $\mathscr{L}^*_b\chi \in \Gamma^+$. In virtue of axiom (A7) and of the closure under *modus ponens* of maximally $\wedge$-consistent sets, $\mathscr{L}_a \in \Gamma^+$. Given that by hypothesis $\mathscr{L}_a\chi \in L_\phi$, this implies that $\mathscr{L}_a\chi \in \Gamma$. (ii) Let us suppose that $\mathscr{L}_a\chi \in \Gamma$; then $a \leq b^*$ hence $P(s_\Gamma,\chi) \geq a$. ■

To prove completeness, we need a last lemma.

Lemma 4. *Let* $At(\phi)$ *the set of atoms in* L_ϕ;

$$At(\phi) = \{\Delta \cap \mathscr{L}_\phi : \Delta \text{ is maximally coherent}\}.$$

Proof. $At(\phi) \subseteq \{\Delta \cap L_\phi : \Delta$ is maximally coherent$\}$ follows from a preceding lemma. Let Γ^+ a maximally consistent set and $\Gamma = \Gamma^+ \cap L_\phi$. We need to show that is maximally consistent in L_ϕ. First Γ is consistent; otherwise, Γ^+ would not be. Then, we need to show that is Γ maximal, i.e. that for every formula $\psi \in L_\phi$, if $\Gamma \cup \{\psi\}$ is consistent, then $\psi \in \Gamma$. Let ψ such a formula. Let us recall that Γ^+ is maximally consistent. Either $\psi \in \Gamma^+$ and then $\psi \in \Gamma$; or $\neg\psi \in \Gamma^+$ (elementary property of maximally consistent sets) and, if $\psi = \neg\chi$, $\chi \in \Gamma^+$ as well. Hence, by definition of L_ϕ, χ or $\neg\neg\chi \in \Gamma$. But this is not compatible with the initial hypothesis according to which $\Gamma \cup \{\phi\}$ is consistent. ■

We can now finish the proof: Let ϕ a *LPM*-consistent formula. Then, there exists a maximally *LPM*-consistent set Γ^+ which contains ϕ. Let $\Gamma = \Gamma^+ \cap \mathrm{L}_\phi$. ϕ is in Γ therefore by the Truth Lemma, ϕ is true in state s_Γ of the ϕ-canonical structure. Then ϕ is satisfiable. ■