Chapter 2

EPISTEMIC MODELS, LOGICAL MONOTONY AND SUBSTRUCTURAL LOGICS*

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2.1 Introduction: logical omniscience and logical monotony

Suppose that a modeller wants to represent the cognitive state (the set of beliefs) of a given reasoner. If this modeller uses a common epistemic logic (i.e., a normal modal logic), even in its weaker form (the so-called system K), his or her model will *necessarily* ascribe to the reasoner a set of beliefs closed under the consequence relation of classical logic. In the literature, this phenomenon is called the problem of logical omniscience (PLO). It is worth noting that this is not an isolated phenomenon: beyond epistemic logic, a vast range of "epistemic models" (that is, formal models of knowledge and belief) like probability theory or belief revision exhibits an analogous form of closure:[1]

$$\frac{A \rightarrow B}{B_i A \rightarrow B_i B}$$

(Rule of Epistemic Monotony)

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[1] In the standard formalisms, $B_i A$ means that the reasoner i believes that A, $L_i^{\alpha} A$ means that the proposition expressed by A has at least probability α for i and $A \in K_i$ means that the proposition expressed by A is in the belief set of i.

J. van Benthem, G. Heinzmann, M. Rebuschi and H. Visser (eds.), The Age of Alternative Logics: Assessing Philosophy of Logic and Mathematics Today, 11–23.

$$\frac{A \rightarrow B}{L_i^\alpha A \rightarrow L_i^\alpha B}$$

(Rule of Probabilistic Monotony)

$$\text{If } \vdash_{CL} A \rightarrow B \text{ then if } A \in K_i, B \in K_i$$

(Monotony of Revision Property)

All these inference rules are valid in the corresponding epistemic models. All express a form of closure that might be called "logical monotony" and all are, therefore, *uneliminable* assumptions of such models. Logical omniscience is then the particular instance of logical monotony in the case of epistemic logic, and its importance comes first from the fact that it is a simple and representative instance of logical monotony. Since the seminal work of [J. Hintikka 1962], lots of solutions have been defended to solve the (PLO),[2] but there is little consensus as to which are the best; there is even more little consensus as to what would be a good solution to (PLO).

The aim of this paper is, following [J. Dubucs 1991] and [J. Dubucs 2002], to defend a family of proof-oriented solutions to the (PLO) starting from a conceptual analysis of the solutions' space, that is, the aim of the paper is to characterize what would be a good solution to (PLO) and then to propose some logics as solutions to (PLO).

The remainder of the paper proceeds as follows. Section 2 puts some constraints on the solutions' space. This results in a criterion of cognitive realism called the Principle of Epistemic Preservation (PEP). In Section 3, I shall claim that two proposals are more adequate to (PEP) than classical epistemic logic (CEL). Those proposals will be discussed in Section 4. I conclude in Section 5.

2.2 Looking for a better epistemic logic: preliminary steps

2.2.1 The core of (PLO)

There exists today a huge family of alternative epistemic logics that have been devised in order to solve the (PLO). They are characterized by the failure of closure under the classical consequence relation. Among them, the two main proposals are the logic of awareness (AEL) and the logic of impossible worlds (IWEL). In the first case, a set of formulas $\mathcal{A}(s)$ is associated to each world s of the state space S and a formula B_iA is true in s iff in every world s' accessible from s A is true and $A \in \mathcal{A}(s)$. In the second case, a set of "impossible worlds" is added. In those worlds, there are no constraints on the valuation of formulas (e.g. in the same impossible world, it is possible that A and $\neg A$ are true, $A \wedge B$ is true but $B \wedge A$ false, etc.) The point is that there cannot exist more powerful solutions to (PLO) because epistemic logics where the deductive ability is weaker cannot exist. With (IWEL) or (AEL), one might represent the beliefs of reasoners who believe in a set Γ of formulas, but who do not believe any logical consequence of Γ.[3] Hence, the difficulty with (PLO) is not to find powerful enough alternative logics, but to find *good* alternative logics. In this section, my purpose is to define what is a good (or a at least a better) epistemic logic compared to the classical one. Many reasons can bring a reasoner not to believe a consequence A of a set of beliefs Γ. At one extreme, it might be that A is a trivial consequence of Γ, but that the reasoner reasons very poorly or does not pay attention; at the other extreme, it might be that there does not exist any systematic procedure to go from Γ to A. Clearly, the second is a more essential reason, whereas the first is more contingent (a Chomskyan linguist would perhaps say an "error of performance"). My first claim is that the aim of a solution to (PLO) is to capture the latter kind of reasons and to abstract from the former one. This point has several consequences.

First of all, an adequate epistemic logic should not only be deductively weaker than (CEL) but should exhibit such a weakening for essential reasons. (AEL) and (IWEL) do not fit this requirement. In (AEL), a reasoner does not believe in a consequence A of Γ only if the formula A is not a formula of which he is aware; in (IWEL), an agent does not believe in a consequence A of Γ only if A is false in some accessible impossible world. And A is false in some accessible impossible world only if the meaning of some logical connective *changes* with respect to the "true" possible worlds. Hence (AEL) and (IWEL) are not good solutions to (PLO) because neither morphological availability (a formula is morphologically available if the reasoner is aware of its existence) nor the changing nature of connectives are likely to be the essential reasons of bounded deductive ability. On the contrary, it seems to me reasonable to assume that reasoners have a minimally correct understanding of logical connectives. This last requirement, admittedly vague for the moment, can be called the Minimal Rationality Requirement (MRR). To sum up, a good solution to (PLO) has to deal with the core of (PLO), that is, it has to concern the ability to draw inferences. A second consequence is that a "good" epistemic logic will still involve a large measure of idealization with respect to the reasoners' actual deductive behaviors. This is not a bad point because the constitutive assumption of epistemic logic is arguably the fact that cognition often fit the logical standard. Hence, the idea of a *base logic*, that is a logic with respect to which an agent is omniscient, should not be rejected, but it is unreasonable to assume that this base logic is, as in (CEL), classical logic.

[2] See [R. Fagin *et al.* 1995], chap. 9.

[3] See [H. Wansing 1990].

2.2.2 The Principle of Epistemic Preservation

How is one to go beyond this diagnosis ? The syntactic format of usual epistemic logics is the axiomatic or "Hilbert-style" format. It is probably heuristically inadequate because the focusing on information processing leads us to see logic as a set of rules of reasoning more than as a body of abstract truth. The first thing to do is then to move from this format to a "rule-based" format like Natural Deduction (ND) or Sequent Calculus (SC). However, at this stage, one might imagine two distinct approaches: a quantitative one and a qualitative one. In the quantitative approach, one keeps the classical rules, but one restricts the cognitive complexity (e.g. the size) of the possible proofs based on them. A brutal way of implementing this approach could be the following one: given a set of beliefs Γ, only the formulas deducible using proof of size smaller than k are ascribed to the agent. This is not the approach that this paper will defend. In the qualitative approach, one scrutinizes the rules themselves. It is precisely this qualitative approach that I would like to investigate here.

The rules are the basic components of the modeller's predictions concerning the reasoners' cognitive behaviour in the sense that given a rule $(r*)$: $A_1, ..., A_n \vdash B$ of the base logic, if the modeller ascribes $A_1, ..., A_n$ to a reasoner, he or she will necessarily ascribe B too. Thus one way to proceed would be to test the cognitive realism of rules separately. The cognitive realism of a rule $(r*)$ is naturally defined by the fact that if a reasoner believes $A_1, ..., A_n$, he's likely to believe B. Hence cognitive realism is defined by a form of epistemic preservation. The leading principle of the qualitative approach is then the

Principle of Epistemic Preservation (PEP). A rule $(r*) : A_1, ..., A_n \vdash B$ satisfies (PEP) iff when reasoners have justifications for $A_1, ..., A_n$, they have a justification for B.

(PEP) is very strong, but one can, at least, retain the minimal requirement that follows from (PEP), namely the

Preservability Requirement (PR). A rule $(r*)$: $A_1, ..., A_n \vdash B$ satisfies (PR) iff when reasoners have justifications for $A_1, ..., A_n$, it is possible for them to have a justification for B.

Are there base logics that would fit those principles better than (CEL)? The next sections attempts to answer this question in the affirmative.

2.3 Two proposals of weak epistemic logics epistemic logic

The aim of this section is to sketch some arguments in order to show that two proposals, intuitionistic epistemic logic (IEL) and linear epistemic logic (LEL), satisfy (PEP) and (PR) - or, at least, that they satisfy (PEP) and (PR) better than (CEL).

2.3.1 First proposal: an intuitionistic epistemic logic (IEL)

The first proposal made to satisfy (PEP) and (PR) is an intuitionistic epistemic logic (IEL), that is an epistemic logic where intuitionistic logic (IL) is the base logic. The main conceptual motivation for this proposal comes from the BHK-interpretation[4] of logical constants: following this interpretation, one may associate an elementary construction to every logical constant. For example, to the conjunction $\wedge$ is associated the operation of pairing because a justification for $A \wedge B$ is constructed by pairing a justification for A and a justification for B. Let us say that an inference rule passes the *BHK-test* if an elementary operation of this kind can be associated to it; the conceptual motivation to adopt an (IEL) comes from the fact that one can see the BHK-test as a first approximation of the Preservability Requirement (PR) since it guarantees the existence of a construction corresponding to every inference rule.

What is the result of this test? A well-known fact is that not every rule of (NK)[5] passes this BHK-test, since classical absurdity rule (ar):

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A}(ar)$$

is rejected. In epistemic terms, the absurdity rule is rejected because no epistemic preservation is guaranteed. It is not because someone has a justification for the fact that $\neg A$ implies $\bot$ that he or she has a justification for the fact that A. Hence, if one eliminates (ar) from (NK), one obtains a logic which is cognitively more "realistic" than (NK). (NJ) is such a logic, thus (NJ) could be a first approximation of (PR). Therefore, one would obtain a more realist epistemic logic by replacing the classical base logic by an intuitionistic one. And the result would be an intuitionistic epistemic logic (IEL).

2.3.2 Second proposal: a linear epistemic logic (LEL)

It is impossible to deny that there is still lots of idealization in the first proposal, for real agents are not omniscient with respect to (IL). Can we do better? Can we find a logic which would be a better approximation than (IEL)? Following [J. Dubucs 1991] and [J. Dubucs 2002], the claim of my second proposal is that a substructural logic like linear logic would be a good candidate.

When one looks for a better approximation of logical competence, the trouble is that in the (ND)-format, it is hard to see how to weaken the base logic without changing the inference rules associated with the connectives, that is, it is hard to see how to weaken the base logic without violating the Minimal Rationality

[4]For Brouwer-Heyting-Kolmogorov, see e.g. [A. Troelstra and D. van Dalen 1988] p. 9.
[5]The usual set of inference rules for (CPL) in (ND).

Requirement of Section 2. But the move to (SC)-format provides an interesting perspective because it permits one to distinguish between different categories of inference rules. More precisely, one can distinguish, on one hand, rules that govern the behavior of logical constants or "logical rules", and, on the other hand, rules that govern the management of the sequent or "structural rules". As [K. Dosen 1993] says, "*a very important discovery in Gentzen's thesis [1935] is that in logic there are rules of inference that don't involve any logical constant.*" What is critical from our point of view is that with such a distinction, by eliminating (or controlling) the structural rules, it is in principle possible to reach a higher level of weakening while keeping the rules for connectives fixed. The main question is then to know whether there are good reasons to think that such rules conflict with (PEP) or (PR).

Among usual structural rules, the most debatable are probably the contraction rule (cr) and the weakening rule (wr). Here are their left-version in (LJ), the Sequent Calculus for (IL):

$$\frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}(cr)$$

$$\frac{\Gamma \vdash B}{\Gamma, A \vdash B}(wr)$$

To evaluate these rules, it is necessary to give an epistemic interpretation of them.

Epistemic Interpretation of (*cr*). One may infer from the fact that the reasoners have a justification for B on the basis of several justifications for A (and other premises) that they have a justification for B on the basis of only one justification for A (and other premises).

Epistemic Interpretation of (*wr*). One may infer from the fact that reasoners have a justification for B on the basis of some premises that they still have a justification for B on the basis on these premises and a new premise A.

Historically, (wr) has been the most challenged of these two rules because it allows a conceptual gap between the premises and the conclusion of a chain of reasoning. The so-called "relevant logics" are designed to correct this point. But if one is focused on the epistemic interpretation of the structural rules, it seems to me that (cr) is the most debatable,[6] and I shall now argue against this rule.

The following argument is a conceptual argument based on the motivation often given for a well-known logic among those that challenge the (unrestricted

[6]This does not exclude the possibility that a rejection of (wr) could be relevant too for (PLO).

use of) contraction rule, namely linear logic (LL). The basic idea is to suggest an intuitive interpretation of logical constants and inference rules in terms of resources and resource-consumption. In this interpretation,

- a formula A is interpreted as a type of resource;
- an occurrence of a formula A in a sequent is a resource of type A; and
- a sequent is interpreted as a relation of resource-consumption.

For example, in [M. Okada 1999], the right rule for the tensor $\otimes$:

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}(tr)$$

is interpreted as meaning that "if A can be generated by using resource Γ and if B can be generated by using resource Δ, then $A \otimes B$ (A and B in parallel) can be generated by using resource Γ and Δ". This interpretation was introduced by [J-Y. Girard 1987] and is systematically developed by [M. Okada 1999]. For this reason, let us call it the "GO-interpretation" to stress the parallelism with the BHK-interpretation mentioned above. And from this interpretation, one can extract a *GO-test* for inference rules similar to the BHK-test described above. The first intermediate step of this argument is that (cr) does not pass the GO-test: if one needs two resources of a given type to do some task, nothing guarantees that, with only one token of this resource, one is able to fulfill a book with two times 10 euros, one cannot infer that this person can buy the same book with only 10 euros; or to take the chemical example of [J-Y. Girard 1995], two molecules of H_2O can be generated by two molecules of H_2 (and one of O_2), but not by one molecule of H_2 (and one of O_2). Generally speaking, (cr) is not valid for a consumption relation.

An important intermediate step is still missing, however, namely: What is the relationship between this GO-test and the epistemic interpretation of (cr)? Looking at these examples from chemistry or book buying, one may note that they focus on *an objective kind of resource-sensitivity*. The question is whether logic can faithfully represent such (objective) processes as book buying or chemical reactions which imply, as noted by [J-Y. Girard 1995], that temporality and especially updating are being taken into account. It is worth noting that this objective resource-sensitivity has in itself nothing to do with a *computational resource-sensitivity*. It is only a matter of making our language and our logic more faithful to an intended interpretation. But, for this reason, this is hardly what we are looking for. We want a more accurate representation of reasoners whose deductive resources (especially their computational power) are limited. Hence we are looking for a *cognitive kind of resource-sensitivity*, as expressed in the epistemic interpretation of (cr). We did not face such a problem with the BHK-test because the BHK-interpretation is arguably intrinsically epistemic whereas the GO-interpretation is not. Consequently, the question is: Does the GO-interpretation make sense in an epistemic context?

I do not have a conclusive answer to this question, but I think the following suggestion is plausible: to fill the gap, we have to see

- a justification for a formula as resource in a reasoning process, and
- a reasoning process as a consumption-relation.

Now I shall develop an (admittedly highly speculative) argument in support of this view. Suppose a reasoner holds the belief that A. This belief has *inferential power* in the sense that a reasoner who believes that A is able to make this belief interact with other beliefs in some reasoning processes. So one can see a reason to believe A as a resource for reasoning processes. But clearly, this inferential power is bounded - otherwise, we would be logically omniscient. Therefore, the inferential power of a belief is a scarce resource. And, precisely, seeing the reasoning process as a consumption-relation permits us to represent this scarcity of inferential power. One can give a more psychological flavour to the rejection of (cr): reasons have a psychological strength and sometimes people would not hold something to be true were they to have fewer reasons for this belief than they actually have. But it seems to me that the fundamental idea is not different. One can speak of "strength" or of "resources". The main point, in both cases, is that a kind of causal power in reasoning is associated to reasons.

2.4 Discussion

This section attempts to answer the main questions and objections raised by the two proposals.

2.4.1 Question 1: Are (IEL) and (LEL) technically possible?

The answer is yes. At the syntactic level, the matter is easy: one has to substitute intuitionistic logic (or linear logic) for classical logic. At the semantic level, things are more complex. In the intuitionistic case, one might take advantage of the well-known Kripke-style semantics for intuitionistic logic. In this semantics, the accessibility relation I is reflexive and transitive and the valuation satisfies a Persistence Property: for every atomic proposition p and every state $s \in [[p]]$,[7] $\forall s'$ s.t. sIs', $s' \in [[p]]$. The basic problem is to keep the Persistence Property when one adds the epistemic modalities. It can be solved either by putting some constraints on I and R[8] (see [M. Bozic and K. Dosen, 1983]) or by changing the satisfaction clause of the modalities. In both cases, the result is a Kripke-style semantics with two accessibility relations. Such a semantics is investigated by a growing literature on intuitionistic modal logic.[9]

In the linear case, a supplementary difficulty comes from the fact that the usual semantics of linear logic is algebraic, and not relational (in the Kripke-style). In order to design a semantics for (LEL), one can therefore either give an algebraic semantics of modal logic or give a relational semantics of linear logic. For example, [M. D'Agostino *et al.* 1997] build a linear modal logic for a simple fragment (implication and modality) by taking the latter approach. Such a semantics is based on a constrained set of states, but the constraints are considerably stronger than they can be in the intuitionistic case. Indeed, the set of spaces has to be a special complete lattice enriched by a binary operator, usually called *quantale*.[10]

2.4.2 Question 2: Are there concrete failures of logical omniscience that could be modelled by (IEL) or (LEL)?

In the previous section, I have defended (IEL) and (LEL) from an abstract point of view. The underlying claim was that choosing a set of beliefs closed by intuitionistic or linear logic is more realistic than choosing a set of beliefs closed by classical logic. But it would be nice to exhibit concrete types of failures of logical omniscience that could be modelled by (IEL) or (LEL). In the intuitionistic case at least, the answer is, not surprisingly, that we can.

Suppose that a reasoner i has a proof that $\neg A$ implies a contradiction, e.g. he or she has a proof that if a continuous function $f : C \times C$ on a n-dimensional simplex C has a fixed point, a contradiction follows. One can then ascribe to him or her the belief $B_i\neg\neg A$. By (ar), it follows that in (CEL), B_iA holds. This will not necessarily be the case in (IEL) since (ar) is not a valid rule. Hence, we will be able to model the common situation where the value x^* s.t. $f(x^*) = x^*$ is not available to the reasoner. That is, the situation where B_iA does not hold. This power of (IEL) can find several applications: in general, it permits one to represent a kind of mathematical ignorance (which is forbidden to (CEL)); in particular, it can be used to model boundedly rational *agents* (reasoners who have to act on the basis of their beliefs) who know that there is a best choice among a given set of possible actions but who do not know how to determine its value.

[7] $[[p]]$ denotes the set of states where p holds.
[8] R denotes the epistemic accessibility relation.
[9] See e.g. [F. Wolter *et al.* 1999].
[10] On those general semantics, cf. [H. Ono 1993].

2.4.3 Question 3: In shifting from (CEL) to (IEL) or (LEL), are we not shifting from one kind of omniscience to another one?

This question raises what can be called "the Big Objection". It is indeed the most common objection made to the kind of approach advocated in this paper, that of proposing a replacement for the base logic. Some remarks have already been made in Section 2. Nevertheless, I now should like to give an extended answer to the Big Objection.

1. Strictly speaking, it is of course correct to say that one is shifting from one sort of omniscience (with respect to classical logic) to another sort of omniscience, but, I repeat, this alone cannot be considered as a sound objection. Why? Because all depends on the consequence relation with respect to which the reasoner will be supposed to be omniscient. If the consequence relation of the new base logic is more realistic than the consequence relation of classical logic, progress has been made. Furthermore, as noted above, with (AEL) or (IWEL), one already knows how to weaken epistemic logic as much as possible; what is important is to find deductively significant weakening of (CEL). Hence, I agree with [R. Fagin *et al.* 1995]: "*It may not be so unreasonable for an agent's knowledge to be closed under logical implication if we have a weaker notion of logical implication.*"

2. However, the previous point is not the core of the Big Objection. Its very core is the following point: is intuitionistic (or linear) omniscience more realistic than classical omniscience? My arguments for an affirmative answer were given in Section 3. But one has to recognize that this is debatable. For example, from a computational point of view, the consequence problem is co-NP-complete in the (propositional) classical case, but PSPACE-complete in the intuitionistic one.[11] Therefore, it seems that (IL) or (LL) are not necessarily more realistic as base logics.

Suppose first that this computational point of view *with respect to the base logics* is relevant. It is not an obvious assumption because while it is clearly relevant concerning the deductive problems that the reasoner faces (e.g., as above, the search for a fixed point of a function), the base logic is above all a model of the reasoner's competence when facing these problems. Then, it is worth noting that the computational point of view is not "univocal" concerning our question: e.g., at the first-order level, the fragment (MALL)[12] of linear logic is decidable and at most NEXPTIME-hard. Hence, proof-oriented weakening does not necessarily increase computational difficulty. What we can conclude is that there is not necessarily a convergence between different criteria of cognitive realism.

[11] Cf. [R. Statman 1979].
[12] Multiplicative Additive Linear Logic: the linear modalities ! and ? do not appear in this fragment.

This is a phenomenon that one can meet elsewhere in the modelling of bounded deductive ability, and even between different kinds measurements of computational complexity, e.g. in the theory of Repeated Games, [C. Papadimitriou 1992] has proved that in the Repeated Prisoner's dilemma, when one limits the space of possible strategies using an upper bound on the size of automata implementing them, the computational complexity of finding a best-response becomes NP-complete.[13]

But is the computational point of view *with respect to the base logics* really relevant? My answer would be less conclusive on this point, but my claim is that it is not really relevant. The reason is this: The main proposal of the paper is not that the agents are reasoning *in* the base logic, but that base logics like (IL) or (LL) promise to fit the reasoners' deductive competence better because they eliminate rules that were unrealistic when interpreted epistemically (that is, interpreted as predictions about the reasoners' justifications, cf. Section 2), e.g. it is not reasonable to suppose that if a reasoner has a justification for $\neg\neg A$, he or she has a justification for A. The computational point of view with respect to the base logic seems, therefore, to confuse the reasoners' level and that of the modeller. Moreover, only an epistemic logic like (IEL) can model computational difficulty, e.g. the fact that reasoners may not be able to find a solution to an instance of the Travelling Salesman Problem whereas they know that such a solution does exist.

To sum up, my answer to the core of the Big Objection is twofold: first, in general, there is no guarantee that the different criteria of cognitive realism are convergent, and it is a difficult challenge to satisfy several of them; second, concerning the computational complexity of the base logic, it is not clear that it is itself a relevant criterion of cognitive realism.

2.5 Conclusion

There are of course many more questions raised by the two proposals made in this paper than those discussed in the previous section. For example, it is well-known that, in the absence of certain structural rules, a phenomenon of *splitting* appears among logical constants. It is important to note that this phenomenon does not in itself violate the Minimal Rationality Requirement since in a (LEL) the logical rules are fixed and well-defined. But, if one uses the expressive power of linear logic (even with additive and multiplicative constants only), one introduces a gap between our ordinary and intuitive grasp of the meaning of logical constants and the logical constants of epistemic logic. On topics like the previous one, the discussion isn't closed. But I would like to conclude by making a more general point.

[13] It is polynomial when the space of strategies is unbounded.

The central Principle of Epistemic Preservation opens the way to a wide spectrum of weak epistemic logics, that is epistemic logics where the consequence relation is weakened.

First, concerning the whole spectrum of weak epistemic logics, this "qualitative approach" has its limits because it does not permit a step-by-step control of inferences processes like, e.g. the proposal of [H. N. Duc 2001], more akin to what I labelled earlier the "quantitative approach", but it has comparative advantages too, e.g. the fact that a true semantics for belief is still possible.

Second, concerning the different logics in the spectrum, it is worth noting that the weaker that one makes the base logic, the less the formal implementation of the corresponding epistemic logic is manageable. The semantics of (CEL) is simpler than the semantics of (IEL), which in turn is simpler than the semantics of (LEL). I do not think that there is any paradox to be found in this fact. One can observe a quite similar phenomenon in decision theory in case of uncertainty where, for reasons of descriptive realism, the (simple) model of Subjective Expected Utility is generalized by non-additive probabilities (in general, much less simple ones), but a loss of simplicity is often considered as the price to be paid for this descriptive gain. From this point of view, (IEL) could, in the short term at least, be a good trade-off between the simplicity of (CEL) and the accuracy of (LEL).

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