Impossible states at work

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introduction

- decision theory has three building blocks :
  - a model of beliefs (doxastic model)
  - a model of desires (axiological model)
  - a criterion of choice, which, given beliefs and desires, selects the "right" actions
- *full beliefs* :
  - the classical model is epistemic logic
  - it is well known that epistemic logic suffer from logical omniscience (LO) : closure of beliefs under logical consequence, substituability of logically equivalent formulas, etc.
  - putative solutions to LO : neighborhood structures, awareness structures, non-standard (impossible) states structures, etc.

introduction, *cont.*

- *partial beliefs*:
  - mainstream decision theory (bayesian decision theory) is based on a model of partial beliefs
  - the classical model of partial beliefs is probabilistic
  - it is less recognized that probabilistic models of beliefs suffer from LO as well
- decision theory inherits the cognitive idealizations of its doxastic models
  $\hookrightarrow$ question : how to weaken LO in models of partial beliefs ?

introduction, *cont.*

- method : extension of the main (putative) solutions to LO in epistemic logic
- main claim : non-standard structures is the best candidate to this extension
- Plan of the talk
  - Section 1 : logical omniscience and epistemic logic
  - Section 2 : implicit probabilistic structures without LO
  - Section 3 : explicit probabilistic structures without LO

sec. 1, epistemic logic

- **Definition** : The set of **formulas** of an epistemic propositional language $\mathcal{L}B(At)$ based on a set At of propositional variables $Form(\mathcal{L}B(At))$, is defined by

$$\phi ::= p|\neg\phi|\phi \vee \psi|B\phi$$

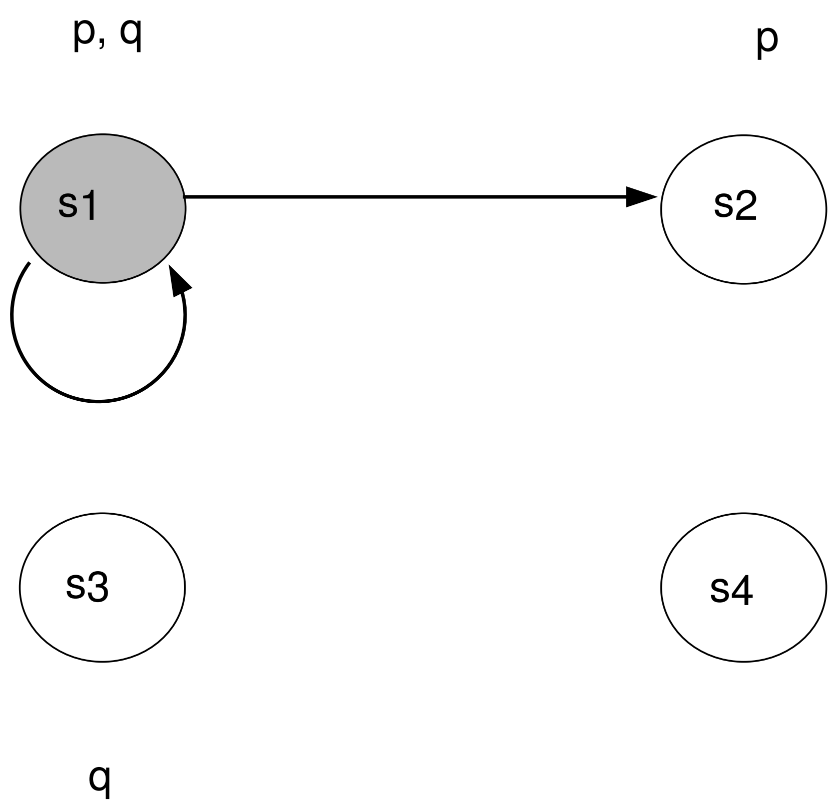
  where $p \in At$.

- **Definition** : let $\mathcal{L}B(At)$ an epistemic propositional language ; a **Kripke structure** for $\mathcal{L}B(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, R)$ where
  (i) S is a state space,
  (ii) $\pi : At \times S \rightarrow \{0, 1\}$ is a valuation
  (iii) $R \subseteq S \times S$ is an accessibility relation

sec. 1, epistemic logic, *cont.*

- **Definition** : $\bar{\pi}$, called the **satisfaction relation**, extends π to every formula of the langage according to the following conditions :
  - $\bar{\pi}(s, p) = \pi(s, p)$ if $p \in At$
  - $\bar{\pi}(s, \phi \vee \psi) = 1$ iff $\bar{\pi}(s, \phi) = 1$ or $\bar{\pi}(s, \psi) = 1$
  - $\bar{\pi}(s, \neg\phi) = 1$ iff $\bar{\pi}(s, \phi) = 0$
  - $\bar{\pi}(s, B\phi) = 1$ iff $\forall s'$ s.t. sRs', $\bar{\pi}(s, \phi) = 1$ (= ***possible-state analysis of belief*** = to believe something is to exclude that it could be false)

sec.1, epistemic logic, example



Pierre believes that p, hence that $p \vee q$.

sec. 1, logical omniscience

- **Definitions** :

  (i) the **proposition** expressed by ϕ, or **informational content** of ϕ is $[[\phi]]_{\mathcal{M}} = \{s : \bar{\pi}(\phi, s) = 1\}$

  (ii) ϕ **$\mathcal{M}$-implies** ψ if $[[\phi]]_{\mathcal{M}} \subseteq [[\psi]]_{\mathcal{M}}$

  (iii) ϕ and ψ are **$\mathcal{M}$-equivalent** if $[[\phi]]_{\mathcal{M}} = [[\psi]]_{\mathcal{M}}$

- **Proposition**: for all Kripke structure $\mathcal{M}$,
  - **Deductive monotony** : if ϕ $\mathcal{M}$-implies ψ, then $B\phi$ $\mathcal{M}$-implies $B\psi$
  - **Intensionality** : if ϕ and ψ are $\mathcal{M}$-equivalent, then $B\phi$ and $B\psi$ are $\mathcal{M}$-equivalent

sec.1, neighborhood structures

- **Definition** : a **neighborhood structure** is a 3-tuple $\mathcal{M} = (S, \pi, V)$ where

  (i) S is a state space,

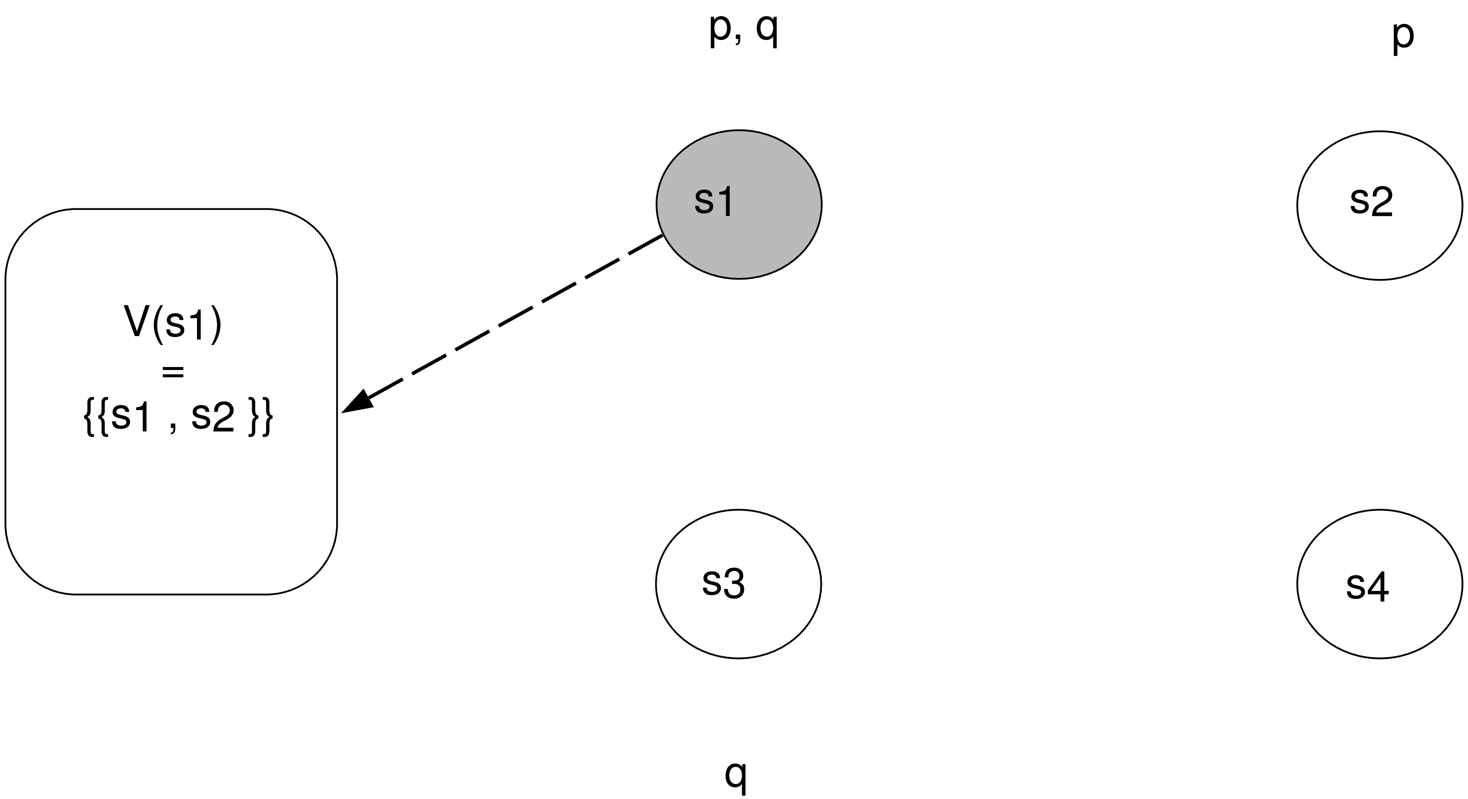
  (ii) $\pi : At \times S \longrightarrow \{0, 1\}$ is a valuation,

  (iii) $V : S \longrightarrow \wp(\wp(S))$, called the agent's **neighborhood system**, associates to every state a set of propositions.

- New doxastic satisfaction condition :

$$\bar{\pi}(B\phi, s) = 1 \text{ iff } [[\phi]]_{\mathcal{M}} \in V(s)$$

sec.1, neighborhood structures, example



Pierre believes that p but not that $p \vee q$ since $p \vee q \notin V(s_1)$.

sec.1, neigborhood structures, axiomatization

System E (Chellas, 1980)

(PROP) Instances of propositional tautologies

(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

(RE) From $\phi \leftrightarrow \psi$ infer $B\phi \leftrightarrow B\psi$

sec.1, awareness structures

- **Definition** : an **awareness structure** is a 4-tuple (S, π, R, A) where

  (i) S is a state space,

  (ii) $\pi : At \times S \rightarrow \{0, 1\}$ is a valuation,

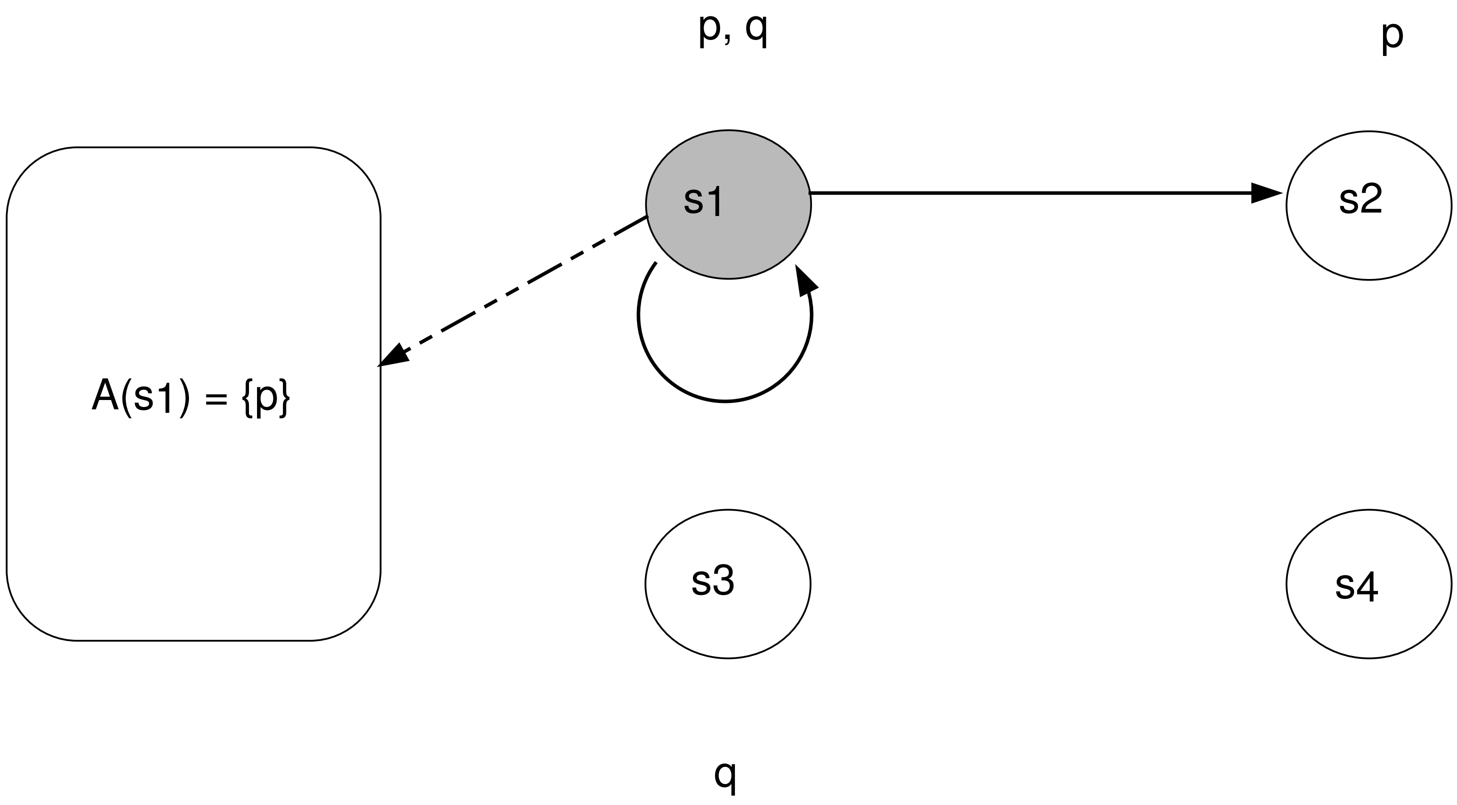
  (iii) $R \subseteq S \times S$ is an accessibility relation,

  (iv) $A : S \rightarrow Form(\mathcal{L}B(At))$ is a function which maps every state in a set of formulas ("awareness set").

- New doxastic satisfaction condition :

$$\bar{\pi}(B\phi, s) = 1 \text{ iff } \forall s' \text{ s.t. } sRs', s' \in [[\phi]]_{\mathcal{M}} \text{ and } \phi \in A(s)$$

sec.1, awareness structures, example



Pierre believes that p but not that $p \vee q$ since $p \vee q \notin A(s_1)$.

sec.1, structure d'awareness, axiomatization

Minimal Epistemic Logic (FHMV 1995)

(PROP) Instances of propositional tautologies

(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

sec.1, non-standard structures

- **Definition** : a **non-standard structure** is a 4-tuple $\mathcal{M} = (S, S', \bar{\pi}, R)$ where

  (i) S is a space of standard states,

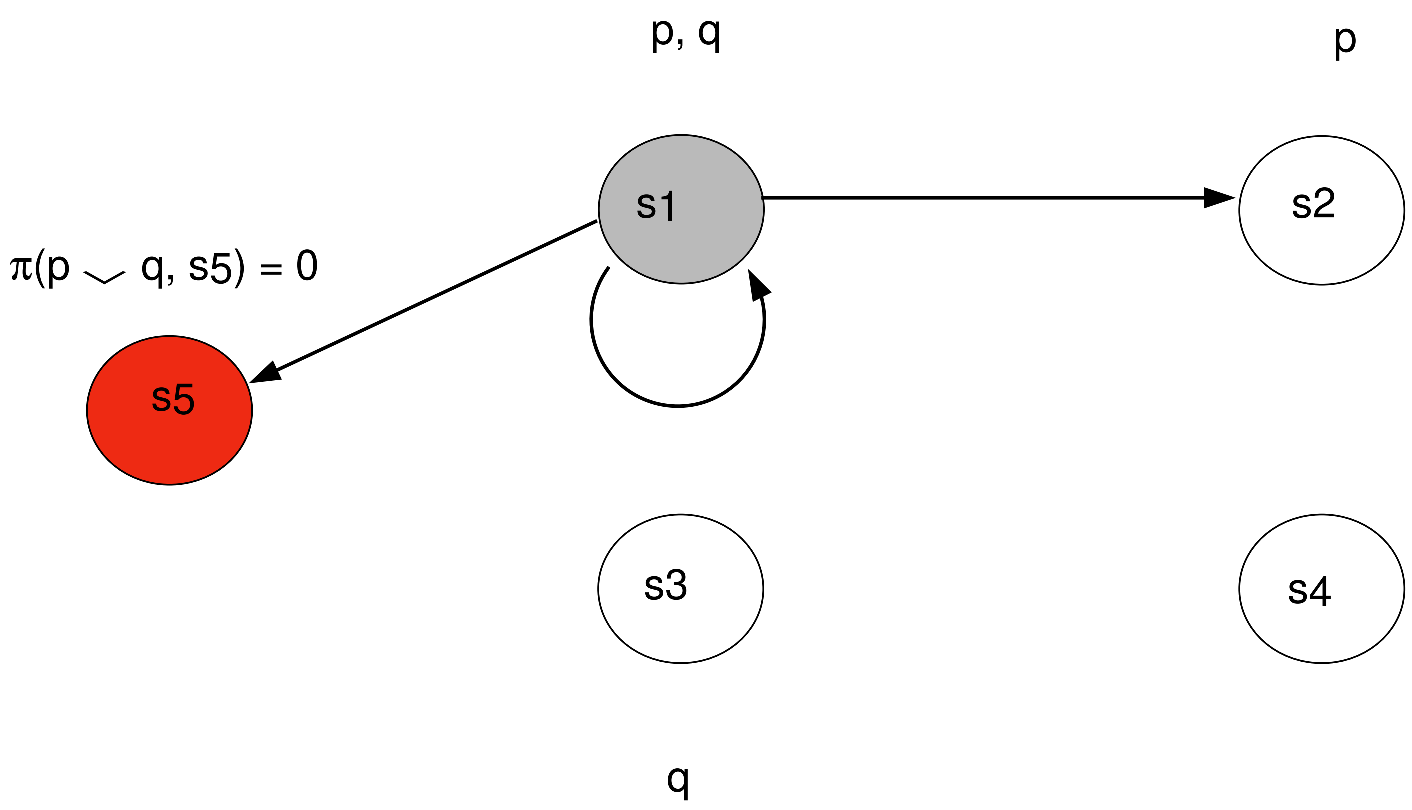
  (ii) S' is a space of non-standard states,

  (iii) $R \subseteq S \cup S' \times S \cup S'$ is an accessibility relation,

  (iv) $\pi : Form(\mathcal{L}B(At)) \times S \rightarrow \{0, 1\}$ is a satisfaction relation **standard on** S

- **Definition** : the subjective informational content of ϕ is the set of states where ϕ is true : $[[\phi]]^*_{\mathcal{M}} = \{s \in S^* : \pi(\phi, s) = 1\}$
- Doxastic satisfaction condition : for $s \in S$, $\pi(s, B\phi) = 1$ iff for all s' s.t. sRs', $s' \in [[\phi]]^*_{\mathcal{M}}$

sec.1, non-standard structures, example



Pierre believes that p but not that $p \vee q$ since $\pi(s_5, p \vee q) = 0$ and $s_1 R s_5$.

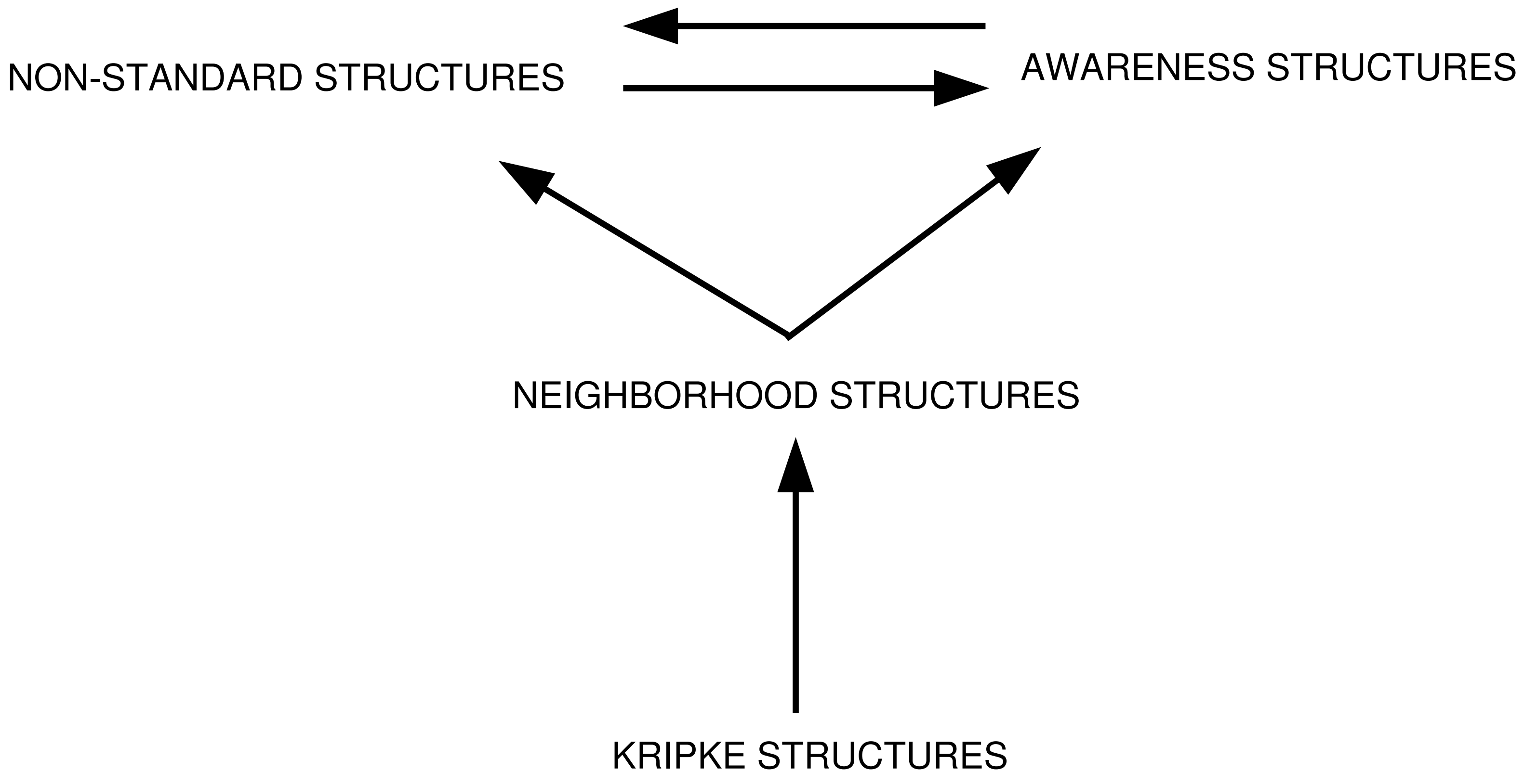
sec.1, non-standard structures, axiomatization

Minimal Epistemic Logic (Wansing 1991)

(PROP) Instances of propositional tautologies

(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

sec.1, respective powers



sec. 2, implicit probabilistic structures

- **Definition** : let $\mathcal{L}(At)$ a propositional language ; an **implicit probabilistic structure** for $\mathcal{L}(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, P)$ where
  (i) S is a state space,
  (ii) π is a valuation,
  (iii) P is a probability distribution on S.
- An agent believes to degree r a formula $\varphi \in Form(\mathcal{L}(At))$, $PB(\varphi) = r$, iff $P([[\varphi]]_{\mathcal{M}}) = r$

sec.2, probabilistic omniscience

- **Proposition** : for all probabilistic structure $\mathcal{M}$,
  - (i) **deductive monotony** : if ϕ $\mathcal{M}$-implies ψ, then $PB(\phi) \geq PB(\psi)$.
  - (ii) **intensionality** : if ϕ and ψ are $\mathcal{M}$-equivalent, then $PB(\phi) = PB(\psi)$.
- *Certainty (r= 1)*
  - (i) if ϕ $\mathcal{M}$-implies ψ, if it is certain that ϕ, it is certain that ψ
  - (ii) if ϕ and ψ are $\mathcal{M}$-equivalent, ϕ is certain iff ψ is certain.

sec.2, non-standard implicit probabilistic structures (NSIPS)

- **Definition** : let $\mathcal{L}(At)$ a propositional language ; a **non-standard implicit probabilistic structure** for $\mathcal{L}(At)$ is a 4-tuple $\mathcal{M} = (S, S', \pi, P)$ where

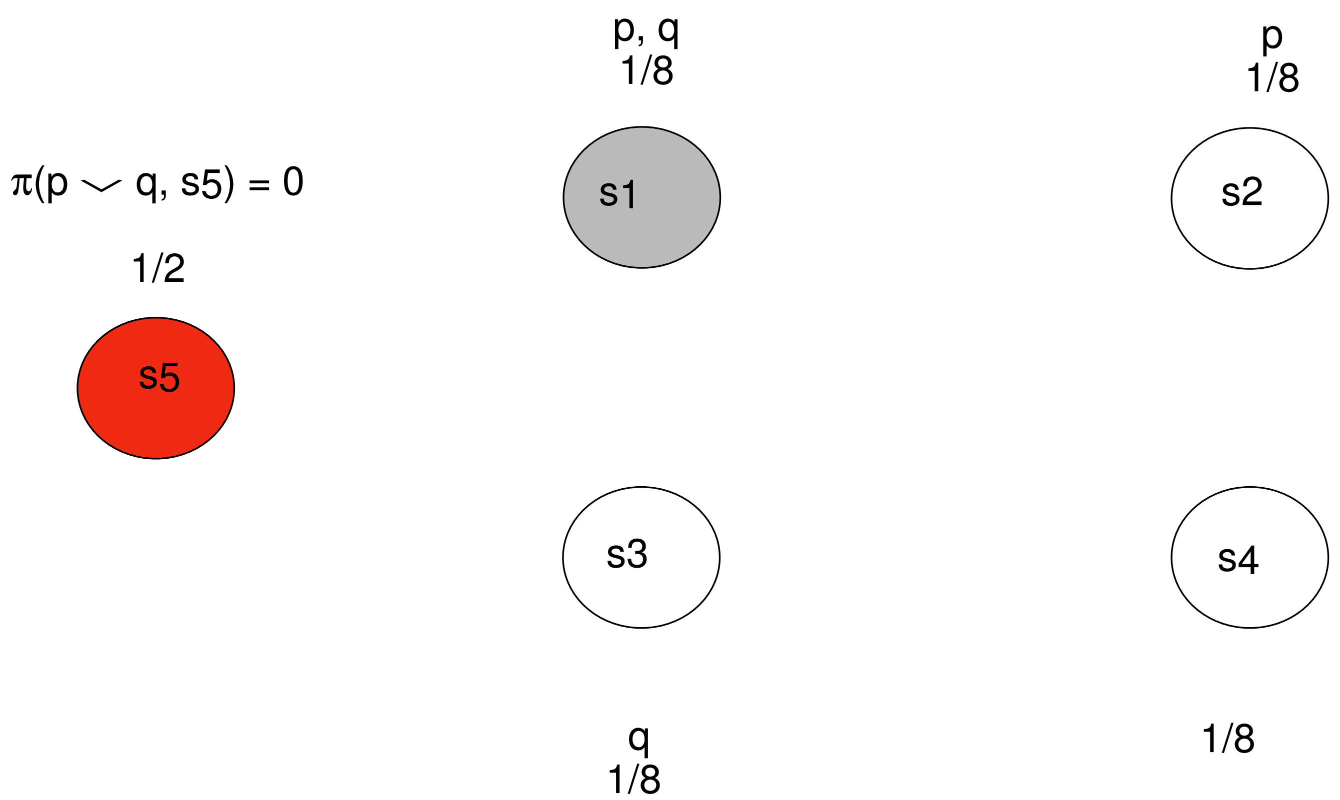
  (i) S is a standard state space,

  (ii) S' is a non-standard state space,

  (iii) $\pi : Form(L(At)) \times S \cup S' \longrightarrow \{0, 1\}$ is a satisfaction relation which is standard on S,

  (iv) P is a probability distribution on $S^* = S \cup S'$

sec.2, NSIPS, example



Pierre believes that p to degree $3/4$, but that $p \vee q$ to degree $3/8$.

sec.2, deductive information and learning in NSIPS

- What does it mean to learn that ϕ implies ψ ?
  According to the possible-state analysis of belief : exclude states where ϕ is true but ψ false ; hence, to learn the event

$$I(\phi, \psi) = S^* - ([[\phi]]^*_{\mathcal{M}} - [[\psi]]^*_{\mathcal{M}})$$

- Compatibility between conditionalization on I and deductive monotony
  Proposition : if $I(\phi, \psi)$ is learned according to Bayes rule, then deductive monotony is regained, *ie* $PB_I(\phi) \leq PB_I(\psi)$.

sec.3, explicit probabilistic structures (EPS)

- **Définition** : the set of formulas of an explicit probabilistic language $\mathcal{L}L(At)$ based on a set At of propositional variables, $Form(\mathcal{L}L(At))$ is defined by :

$$\phi ::= p|\neg\phi|\phi \vee \psi|L_a\phi$$

where $p \in At$ and $a \in [0,1] \subseteq \mathbb{Q}$.

- **Définition** : an **explicit probabilistic structure** for $\mathcal{L}L_a(At)$ is a 3-tuple $\mathcal{M} = (S, \pi, P)$ where $P : S \rightarrow \Delta(S)$ assigns to every state a probability distribution on the state space.
- Satisfaction condition for L_a : $\bar{\pi}(s, L_a\phi) = 1$ iff $P(s)([[\phi]]) \geq a$

sec.3, EPS, *cont.*

- Remark 1 : this language is the one proposed by Aumann 1999 and Heifetz and Mongin 2001. Fagin, Halpern and Meggido 1990 and Halpern 2003 use a different language.
- Remark 2 : the EPS are developed by game-theorists because it correspond to the *type spaces* used in games of incomplete information, in the same way that Kripke structures (with R as an equivalence relation) corresponds to the information partitions
- Remark 3: from the explicit probabilistic language, one can define
  - $M_a\phi$ - the agent believes at most to degree a that ϕ - as $L_{1-a}\neg\phi$
  - $E_a\phi$ - the agent believes exactly to degree a that ϕ - as $M_a\phi \wedge L_a\phi$

sec.3, EPS, axiomatization

System HM (Heifetz and Mongin 2001)

(PROP) Instances of propositional tautologies

(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ

(RE) From $\phi \leftrightarrow \psi$ infer $L_a\phi \leftrightarrow L_a\psi$

(A1) $L_0\phi$

(A2) $L_a\top$

(A5)$L_a\phi \rightarrow \neg L_b\neg\phi \ (a + b > 1)$

(Def M) $M_a \leftrightarrow L_{1-a}\neg\phi$

(A8)$\neg L_a\phi \rightarrow M_a\phi$

(B) From $((\phi_1, ..., \phi_m) \leftrightarrow (\psi_1, ..., \psi_n))$ infer
$(\neg M_{a1}\phi_1 \quad \wedge \quad (\bigwedge_{i=2}^{m} L_{ai}\phi_i) \quad \wedge \quad (\bigwedge_{j=2}^{n} M_{bj}\psi_j) \quad \rightarrow$
$\neg M_{(a1+...+am)-(b1+...+bn)}\psi_1)$

sec.2, EPS, axiomatization, cont.

- some theorems :

  (RN) From $\phi \rightarrow \psi$ infer $L_a\phi \rightarrow L_a\psi$

  (A2+) $\neg L_a \bot$ $(a > 0)$

  (A3) $L_a(\phi \wedge \psi) \wedge L_b(\phi \wedge \neg\psi) \rightarrow L_{a+b}\phi$ $(a + b \leq 1)$

  (A4)$\neg L_a(\phi \wedge \psi) \wedge \neg L_b(\phi \wedge \neg\psi) \rightarrow \neg L_{a+b}\phi$ $(a + b \leq 1)$

  (A7) $L_a\phi \rightarrow L_b\phi$ $(b < a)$

sec.3, non-standard explicit probabilistic structures (NSEPS)

- **Définition** : an **non-standard explicit probabilistic structure** for $\mathcal{L}L_a(At)$ is a 4-tuple $\mathcal{M} = (S, S'\pi, P)$ where π is standard on S and $P : S^* \longrightarrow \Delta(S^*)$ assigns to every state a probability distribution on the state space.

  Remark : $\top$ $\bot$ are added with the meanings :

  - $\top$ is what the agent recognizes as necessarily true, then for every $s \in S^*$, $\pi(s, \top) = 1$
  - $\bot$ is what the agent recognizes as necessarily false, then for no $s \in S^*$, $\pi(s, \bot) = 1$

sec.3, NSEPS, axiomatization

Minimal Probabilistic Logic
(PROP) Instances of propositional tautologies
(MP) From ϕ and $\phi \rightarrow \psi$ infer ψ
(A1) $L_0\phi$
(A2) $L_a\top$
(A2+) $\neg L_a\bot$ $(a > 0)$
(A7) $L_a\phi \rightarrow L_b\phi$ $(b < a)$

Theorem : $\models_{NSEPS} \phi$ ssi $\vdash_{MPL} \phi$

sec.3, axiomatization, *cont.*

- Proof of completeness :
  - method of canonical models with filtration
  - for each formula ϕ, one defines a subset of formulas ; in this subset of formulas, *atoms* are maximal coherent set of formulas
  - the hard stuff is to define a canonical probability distribution
  - idea : let Γ be an atom ; in the (standard state) s_Γ, $P(s_\Gamma)$ is an equiprobability on non-standard states s.t. for every formula χ s.t. some $L_a\chi$ are in Γ, a proportion b^* of non-standard states make χ true, where $b^* = \max_b L_b\chi \in \Gamma$.

conclusion

- it is only a first step from the decision theorist's point of view
- the next step is to provide a *representation theorem* à la Savage : to find axioms on the agent's preferences s.t. there exists a non-standard structure and an utility function that rationalize these preferences
- recent tentative by B.Lipman 1999